

AUTOMORPHISMS OF THE FUNDAMENTAL GROUP OF A CLOSED, ORIENTABLE 2-MANIFOLD¹

JOAN S. BIRMAN

In the following pages we note a relationship which exists between the mapping class group (or homeotopy group) of a closed, orientable 2-manifold and Artin's braid group. Let T_g be a closed, orientable 2-manifold of genus g , and let $\pi_1 T_g$ be its fundamental group. Let $M(T_g)$ be the mapping class group of T_g , that is the group of all isotopy classes in the space of all orientation-preserving homeomorphisms of $T_g \rightarrow T_g$, or equivalently [4] the group $\text{Aut } \pi_1 T_g / \text{Inn } \pi_1 T_g$. Let B_n be the Artin braid group, that is the abstract group on $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$ with defining relations

$$\begin{aligned} \sigma_i &\leftrightarrow \sigma_j, & i = 1, 2, \dots, n-1, & \quad |i-j| \geq 2, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & i = 1, 2, \dots, n-2. \end{aligned}$$

The group $M(T_1)$ is the modular group, which is a homomorphic image of B_3 and also (with an appropriate choice of generators) of B_4 . It was shown by Bergau and Mennicke [1] that $M(T_2)$ is a homomorphic image of B_5 . We are able to generalize these known results to show that $M(T_g)$ contains a sequence of $(n-1)$ -generator subgroups which are homomorphic images of B_n for $n=3, 4, \dots, 2g+2$, the union of these subgroups generating all of $M(T_g)$. If $g > 2$, all of these subgroups are proper in $M(T_g)$.

PROOF. Generators for $M(T_g)$ are known, and we use a particularly simple set which was discovered by M. Dehn [2] and recently rediscovered and simplified by W. B. R. Lickorish [3]. Let A be an annulus in the Euclidean plane, parametrized by (r, θ) , where

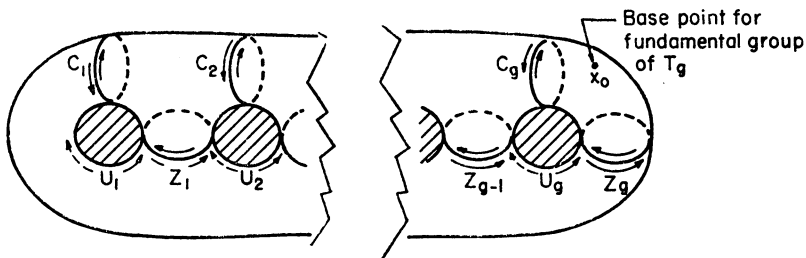


FIGURE 1

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$1 \leq r \leq 2$ and θ is a real number mod 2π . We define a homeomorphism $h: A \rightarrow A$ by $h(r, \theta) = (r, \theta - 2\pi r)$. If C is now any simple closed curve on T_θ , and $e_C: A \rightarrow T_\theta$ an imbedding of $A \rightarrow$ a neighborhood of C in T_θ , which maps $r = 1.5$ onto C , then $e_C h e_C^{-1} / e_C A$ may be extended by the identity map on $(T_\theta - e_C A)$ to a homeomorphism $h_C: T_\theta \rightarrow T_\theta$, which is called a twist about C . It was shown in [3] that the twists $\{h_{U_i}, h_{Z_i}, h_{C_j}; 1 \leq i \leq g, 1 \leq j \leq g-1\}$ about the curves $\{U_i, Z_i, C_j\}$ in Figure 1 generate $M(T_\theta)$. The arrows indicate the direction of relative motion of points on T_θ when each such twist is performed.

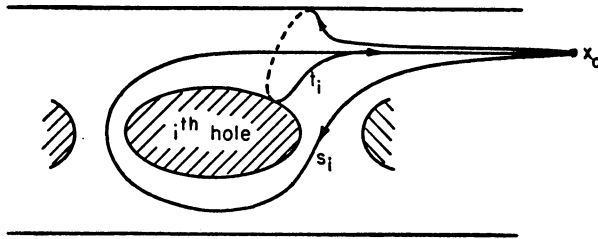


FIGURE 2

Each of these twists induces an automorphism of $\pi_1 T_\theta$, and these can be determined explicitly as follows. We take a base point for $\pi_1 T_\theta$ as indicated in Figure 1, and define generators $\{t_i, s_i; i = 1, 2, \dots, g\}$ for $\pi_1(T_\theta, x_0)$ which are represented by the loops illustrated in Figure 2. In terms of these generators the group $\text{Aut } \pi_1 T_\theta$ has the single defining relation

$$\prod_{i=1}^g s_i^{-1} \bar{t}_i s_i \bar{t}_i^{-1} = 1$$

where

$$\bar{t}_1 = s_1 t_1 s_1^{-1}, \bar{t}_i = s_i (\bar{t}_{i-1} \bar{t}_{i-1}^{-1} t_i) s_i^{-1} \text{ if } i \neq 1.$$

The automorphism induced by a typical twist $h_{Z_{g-1}}$ on the t_i, s_i will now be determined. Clearly $h_{Z_{g-1}}$ will not effect any of the t_i , nor will

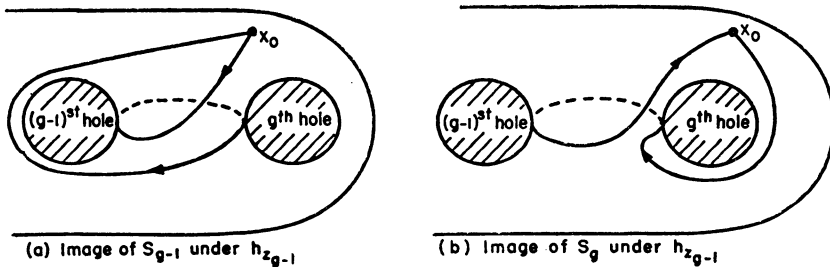


FIGURE 3

$h_{Z_{g-1}}$ effect s_i if $i \neq g$ or $g-1$. To determine the effect of $h_{Z_{g-1}}$ on s_{g-1} and s_g , we observe that, under the action of $h_{Z_{g-1}}$: s_{g-1} and s_g will be altered to the curves illustrated in Figures 3a and 3b respectively, which can be seen to be homotopic to the curves $t_{g-1}^{-1}t_g s_{g-1}$ and $s_g t_g^{-1} t_{g-1}$ on T_g . The induced automorphisms for all possible cases can be determined from similar pictures, and are found to be:

$$\begin{aligned}
 h_{U_i}: t_i &\rightarrow t_i s_i \\
 h_{C_i}: s_j &\rightarrow t_i s_j t_i^{-1} && \text{if } j < i \\
 &t_j \rightarrow t_i t_j t_i^{-1} && \text{if } j < i \\
 &s_i \rightarrow s_i t_i^{-1} \\
 h_{Z_i}: s_i &\rightarrow t_i^{-1} t_{i+1} s_i && \text{if } i = 1, 2, \dots, g-1 \\
 &s_{i+1} \rightarrow s_{i+1} t_{i+1}^{-1} t_i && \text{if } i = 1, 2, \dots, g-1 \\
 h_{Z_g}: s_g &\rightarrow t_g^{-1} s_g
 \end{aligned}$$

where it is understood that every generator which is not listed explicitly is unaltered by the h 's.

Since $\{h_{U_i}, h_{Z_i}, h_{C_j}; 1 \leq i \leq g, 1 \leq j \leq g-1\}$ generate $M(T_g)$, the automorphisms listed above generate $\text{Aut } \pi_1 T_g / \text{Inn } \pi_1 T_g$. This provides a very simple tool for verifying relations in $M(T_g)$: one simply calculates the automorphism induced by a word one suspects is a relator, and observes whether this automorphism is an inner automorphism of $\pi_1 T_g$. Using this procedure, we find the following relations hold among the $h_{U_i}, h_{Z_i}, h_{C_j}$.

- (1) $h_{U_i} \Leftrightarrow h_{U_j}, \quad i \neq j,$
- (2) $h_{C_i} \Leftrightarrow h_{C_j}, \quad i \neq j,$
- (3) $h_{Z_i} \Leftrightarrow h_{Z_j}, \quad i \neq j,$
- (4) $h_{U_i} \Leftrightarrow h_{C_j}, \quad i \neq j,$
- (5) $h_{Z_i} \Leftrightarrow h_{C_j}, \quad i, j = 1, 2, \dots, g,$
- (6) $h_{U_i} \Leftrightarrow h_{Z_j}, \quad i \neq j-1 \text{ or } j,$
- (7) $h_{U_i} h_{Z_i} h_{U_i} = h_{Z_i} h_{U_i} h_{Z_i}, \quad i = 1, 2, \dots, g,$
- (8) $h_{U_{i+1}} h_{Z_i} h_{U_{i+1}} = h_{Z_i} h_{U_{i+1}} h_{Z_i}, \quad i = 1, 2, \dots, g-1,$
- (9) $h_{C_i} h_{U_i} h_{C_i} = h_{U_i} h_{C_i} h_{U_i}, \quad i = 1, 2, \dots, g,$
- (10) $(h_{C_1} h_{U_1} h_{Z_1} h_{U_2} h_{Z_2} \dots h_{U_g} h_{Z_g})^{2g+2} = 1,$
- (11) $(h_{C_1} h_{U_1} h_{Z_1} h_{U_2} h_{Z_2} \dots h_{U_g} h^2 h_{U_g} \dots h_{Z_g} h_{U_g} h_{Z_g} h_{U_1} h_{C_1})^2 = 1.$

We cannot say whether this system is complete.

It then follows that there is a homomorphism $\phi: B_{2g+2} \rightarrow M(T_g)$ defined by $\phi(\sigma_1) = h_{C_1}, \phi(\sigma_{2j}) = h_{U_j}, \phi(\sigma_{2j+1}) = h_{Z_j},$ for $j = 1, 2, \dots, g.$

Moreover, since the center of B_n is known to be infinite cyclic and generated by $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$, in view of relation (10) we have that the center of $B_{2g+2} \subset \ker \phi$. This is not, however, the full kernel because relation (11) also holds in $M(T_g)$, and from previous studies of the group B_n it is known that relation (11) is not a consequence of the others in B_n/center .

Moreover, the $(2g-2i+3)$ -generator subgroups of $M(T_g)$ generated by $\{h_{C_i}, h_{U_i}, h_{Z_i}, h_{U_{i+1}}, h_{Z_{i+1}}, \dots, h_{U_g}, h_{Z_g}\}$ are homomorphic images of $B_{2g-2i+4}$ for each $i=2, \dots, g$ under the homomorphism $\phi_i: B_{2g-2i+4} \rightarrow M(T_g)$ defined by $\phi_i(\sigma_1) = h_{C_i}$, $\phi_i(\sigma_{2j}) = h_{U_{j+i-1}}$, $\phi_i(\sigma_{2j+1}) = h_{Z_{j+i-1}}$ for $j=1, 2, \dots, g-i+1$.

Finally, the $(2g-2i+2)$ -generator subgroups of $M(T_g)$ which are generated by $\{h_{U_i}, h_{Z_i}, h_{U_{i+1}}, h_{Z_{i+1}}, \dots, h_{Z_g}\}$ are homomorphic images of $B_{2g-2i+3}$ for each $i=1, 2, \dots, g$ under the homomorphism ψ_i defined by $\psi_i(\sigma_{2j-1}) = h_{U_{j+i-1}}$, $\psi_i(\sigma_{2j}) = h_{Z_{j+i-1}}$ for $j=1, 2, \dots, g-i+1$.

REFERENCES

1. P. Bergau and J. Mennicke, *Über topologische Abbildungen der Brezelsfläche vom Geschlecht 2*, Math. Z. **74** (1960), 414–435.
2. M. Dehn, *Die Gruppe der Abbildungsklassen*, Acta Math. **69** (1938), 135–206.
3. W. B. R. Lickorish, *A finite set of generators for the homeotopy group of a 2-manifold*, Proc. Cambridge Philos. Soc. **60** (1964), 769–778.
4. Wilhelm Magnus, Abraham Karass and Donald Solitar, *Combinatorial group theory*, Interscience, New York, 1966, p. 75.

COURANT INSTITUTE, NEW YORK UNIVERSITY