SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

THE MINIMAX PRINCIPLE AND UNIQUENESS OF THE FRIEDRICHs EXTENSION

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If $S$ is a symmetric operator in a Hilbert space $\mathcal{H}$, and is bounded below, then the Friedrichs extension $T$ of $S$ (cf., for instance, Riesz and Nagy [2, pp. 327–333]) has the same lower bound as $S$. This property does not characterize $T$ uniquely except in special cases, but if $T$ has a purely discrete spectrum, the following holds:

**Theorem.** Let $T'$ be a selfadjoint extension and $T$ the Friedrichs extension of $S$. Assume that both have compact resolvent, and let $(\lambda_n')_{n\in\mathbb{N}}$ resp. $(\lambda_n)_{n\in\mathbb{N}}$ be the eigenvalues for $T'$ and $T$, nondecreasingly ordered, and appearing with appropriate multiplicities. If $\lambda_n' = \lambda_n$ for some $n$, then $T'$ and $T$ have a common eigenvector corresponding to this eigenvalue. If $\lambda_n' = \lambda_n$ for all $n$, then $T' = T$.

The first assertion of this theorem is a simple consequence of the following slight elaboration of the well-known minimax principle (the terminology being that of Kato [1]):

Let $t$ be the closed quadratic form associated with the selfadjoint semi-bounded operator $T$. Let $M$ be a subspace of $\mathcal{H}$ of codimension $n - 1$, and assume that

$$\min\{t[v, v] | v \in \mathcal{M} \cap D(t), \|v\| = 1\} = \lambda_n.$$

Then $M$ contains an eigenvector for $T$ corresponding to the eigenvalue $\lambda_n$.

The second assertion follows from the first when it is observed that a subspace invariant under both $T$ and $T'$ reduces both of these as well as the corresponding quadratic forms.

Observe in conclusion that if $\lambda_n' = \lambda_n$ for all but a finite number of $n$'s, then $T' = T$, for then the quadratic forms have the same domain, and hence are identical.

**Added in Proof.** K. Jörgens has called the author's attention to the reference Berkowitz [3], and pointed out that the theorem remains true with $\lambda_k$ defined as in that paper provided $\lambda_1, \lambda_2, \ldots, \lambda_n$

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are eigenvalues for $T$. Jörgens also suggested that it be mentioned that one may have $\lambda_n' = \lambda_n$ for infinitely many $n$, but $T' \neq T$ even if $S = -d^2/dx^2$ on $C_0^\infty(0, 1)$.

References


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