Let $\Sigma$ be a class of groups. Define the local operator $L$ as follows:

(i) $L^0(\Sigma) = \Sigma$.

(ii) If $\alpha > 0$ is an ordinal number, then $L^\alpha(\Sigma)$ is the class of all groups having an upper-directed cover of subgroups, each belonging to the class $\bigcup \{ L^\beta(\Sigma) \mid \beta < \alpha \}$.

We will consider all classes of groups to be isomorphism-closed. $L^1(\Sigma)$ is the local system defined in [1, p. 166]. It is well known that if $\Sigma$ is closed under the taking of subgroups then $L^2(\Sigma) = L^1(\Sigma)$.

In the following, for each ordinal $\alpha$ of cardinality $\leq \aleph_0$, the continuum, a class of groups will be displayed whose local sequence does not become stationary before $\alpha$ iterations.

First define an equivalent operator for sets: Let $T$ be a set of sets. Define the operator $C$ as follows:

(i) $C^0(T) = T$.

(ii) If $\alpha > 0$ is an ordinal number, then $C^\alpha(T)$ is the set of all sets having an upper-directed cover of subsets, each belonging to the set $\bigcup \{ C^\beta(T) \mid \beta < \alpha \}$.

If $S$ is a set, denote its power set by $P(S)$; if $T$ is a set of sets, we will sometimes call $\bigcup T$ the “underlying set” of $T$.

For any set of sets $\Gamma$ and ordinal $\alpha$, we have that $C^\alpha(\Gamma) \subseteq P(\bigcup \Gamma)$. Thus all such set-theoretic sequences must eventually become stationary, and we may define $|\Gamma|$ to be the smallest ordinal such that $C^{\alpha+1}(\Gamma) = C^\alpha(\Gamma)$.

We wish first to solve the set-theoretic problem by displaying, for any ordinal $\alpha$, a set of sets $T$ satisfying $|T| = \alpha$ (Lemma 1). After the following definitions, a proposition to be used in Lemma 1 will be proved.

**Definition.** Suppose for each $\alpha \in A$, $T_\alpha$ is a set of sets. Define $\Sigma \{ \Gamma_\alpha \mid \alpha \in A \} = \{ f(A) \mid f : A \rightarrow \bigcup \{ \Gamma_\alpha \mid \alpha \in A \} \}$ is a function such that $\forall \alpha \in A, f(\alpha) \in \Gamma_\alpha$. That is, an element of $\Sigma \{ \Gamma_\alpha \mid \alpha \in A \}$ is a union of sets, one chosen from each $\Gamma_\alpha$.

**Definition.** Suppose $\Gamma$ is a set of sets and $S \in C(\Gamma)$ for some ordinal $\alpha$. Thus $S$ possesses an upper-directed cover $\{ X_\mu \mid \mu \in M \}$ of sub-
sets such that each \( X_\alpha \subseteq C^\delta(\Gamma) \) for some \( \beta < \alpha \). We will say that the cover \( \{ X_\alpha \mid \beta \in M \} \) is "augmented" if \( \exists \mu \in M \) such that \( X_\mu \subseteq \Gamma \).

It is easy to see that any \( S \subseteq C^\delta(\Gamma) \), for any \( \alpha \), possesses an augmented cover.

**Proposition.** Suppose for each \( \alpha \in A \), \( \Gamma_\alpha \) is a set of sets such that the collection \( \{ U_\alpha \mid \alpha \in A \} \) of underlying sets is pairwise disjoint. Let \( \Gamma = \sum \{ \Gamma_\alpha \mid \alpha \in A \} \). Then (i) for any ordinal \( \beta, C^\delta(\Gamma) = \sum \{ C^\delta(\Gamma_\alpha) \mid \alpha \in A \} \), and (ii) \( |\Gamma| = \sup \{ |\Gamma_\alpha| \mid \alpha \in A \} \).

**Proof.** Proof of (i). If \( \beta = 0 \), the assertion is immediate. Suppose \( \beta > 0 \) and \( \forall \rho < \beta, C^\delta(\Gamma) = \sum \{ C^\delta(\Gamma_\alpha) \mid \alpha \in A \} \).

Assume \( B \subseteq \sum \{ C^\delta(\Gamma_\alpha) \mid \alpha \in A \} \), so that \( B = \cup f(A) \), where \( f: A \to \cup \{ C^\delta(\Gamma_\alpha) \mid \alpha \in A \} \) is a function such that \( \forall \alpha \in A, f(\alpha) \subseteq C^\delta(\Gamma_\alpha) \).

Thus each \( f(\alpha) = B_\alpha \) has an upper-directed, augmented cover \( \{ X^\delta_\alpha \mid \psi \in I_\alpha \} \) of subsets \( X^\delta_\alpha \), where each \( X^\delta_\alpha \) is a member of \( C^\delta(\Gamma_\alpha) \) for some \( \rho < \beta \). Since the covers are augmented, for each \( \alpha \) let \( X^\delta_\alpha \subseteq \Gamma_\alpha \). Let \( Y_0 = \cup \{ X^\delta_\alpha \mid \alpha \in A \} \); thus \( Y_0 \subseteq \sum \{ \Gamma_\alpha \mid \alpha \in A \} \). For each \( \rho < \beta \) define \( \Lambda_\rho \subseteq \sum \{ C^\delta(\Gamma_\alpha) \mid \alpha \in A \} = C^\delta(\Gamma) \) as follows:

\[
Y = \Lambda_\rho \Leftrightarrow Y = \cup \{ X^\delta_\alpha \mid \alpha \in A \}, \text{ where } X^\delta_\alpha \subseteq C^\delta(\Gamma_\alpha)
\]

and for all but a finite number of \( \alpha, X^\delta_\alpha = X^\delta_\beta \).

Thus if we put \( \Lambda = \cup \{ \Lambda_\rho \mid \rho < \beta \} \subseteq \cup \{ C^\delta(\Gamma) \mid \rho < \beta \} \), in order to show \( B \subseteq C^\delta(\Gamma) \), it suffices to show that \( \Lambda \) is an upper-directed cover for \( B \).

To show that \( \Lambda \) covers \( B \), select any \( b \in B \). Then \( \exists \alpha', \psi' \) such that \( b \in X^\delta_\alpha' \), since \( \{ X^\delta_\alpha \mid \alpha \in A, \psi \in I_\alpha \} \) covers \( B \). Also \( \exists \rho < \beta \) such that \( X^\delta_\alpha \subseteq C^\delta(\Gamma_\alpha) \). Thus if we let

\[
Y = \cup \{ X^\delta_\alpha \mid \alpha \in A \}, \text{ where } X^\delta_\alpha = X^\delta_\alpha' \text{ when } \alpha = \alpha', \text{ but } X^\delta_\alpha = X^\delta_\alpha' \text{ when } \alpha \neq \alpha',
\]

then \( Y \subseteq \Lambda \cap \Lambda \) and \( b \subseteq Y \). Thus \( \Lambda \) covers \( B \).

To show that \( \Lambda \) is upper-directed, let \( Y_1, Y_2 \subseteq \Lambda \). Say \( Y_1 = \cup \{ X^\delta_\alpha \mid \alpha \in A \} \) and \( Y_2 = \cup \{ X^\delta_\alpha \mid \alpha \in A \} \). Let \( \alpha_1, \ldots, \alpha_n \) be elements of \( A \) such that \( \forall \alpha \in A, X^\delta_\alpha \subseteq X^\delta_{\alpha_1} = X^\delta_\alpha' \) and \( X^\delta_{\alpha_n} = X^\delta_\alpha \), such a set \( \{ \alpha_1, \ldots, \alpha_n \} \) exists by the finiteness condition in the definition of \( \Lambda \). Since \( \forall \alpha \in A, \{ X^\delta_\alpha \mid \psi \in I_\alpha \} \) is an upper-directed cover of \( B_\alpha \), we have that for each \( \alpha_i, 1 \leq i \leq n, \exists \psi_i \in I_{\alpha_i} \) such that \( X^\delta_{\alpha_i} \subseteq X^\delta_{\alpha_i'} \) and \( X^\delta_{\alpha_i} \subseteq X^\delta_{\alpha_i} \). Also \( \exists \rho < \beta \) such that \( X^\delta_{\alpha_i} \subseteq C^\delta(\Gamma_{\alpha_i}) \). Letting \( \rho' = \max \{ \rho_i \mid 1 \leq i \leq n \} < \beta \), we have \( Y = \cup \{ X^\delta_{\alpha_i} \mid 1 \leq i \leq n \} \cup \{ X^\delta_\alpha \mid \alpha \in A \} \subseteq \Delta \subseteq \Lambda \), and \( Y_1 \subseteq Y, Y_2 \subseteq Y \). Thus \( \Lambda \) is upper-directed.

It follows that \( B \subseteq C^\delta(\Gamma) \), and hence \( \sum \{ C^\delta(\Gamma_\alpha) \mid \alpha \in A \} \subseteq C^\delta(\Gamma) \).
Now assume $B \in C^\beta(T)$. Then $B$ possesses an upper-directed cover \{ $Y_\alpha \mid \alpha \in M$ \} where each $Y_\alpha$ is an element of $C^\rho(\Gamma)$ for some $\rho_\alpha < \beta$. Thus $Y_\alpha \in \Sigma \{ C^\rho(\Gamma_\alpha) \mid \alpha \in A \}$, and we may write

$$Y_\alpha = \bigcup \{ X_\alpha \mid \alpha \in A \}, \text{ where } X_\alpha \in C^\rho(\Gamma_\alpha).$$

By the disjointness property of the sets underlying the $\Gamma_\alpha$, we have that $B_\alpha = B \cap (\bigcup \Gamma_\alpha) = \bigcup \{ X_\alpha \mid \alpha \in M \}$ and, since $Y_\alpha \in M$ is upper-directed, $\{ X_\alpha \mid \alpha \in M \}$ is also. Thus $\forall \alpha \in A$, $B_\alpha \in C^\beta(\Gamma_\alpha)$ and $B = \bigcup \{ B_\alpha \mid \alpha \in A \}$. It follows that $B \in \Sigma \{ C^\beta(\Gamma_\alpha) \mid \alpha \in A \}$, and hence $C^\beta(T) \subseteq \Sigma \{ C^\beta(\Gamma_\alpha) \mid \alpha \in A \}$.

The proof of (i) is now complete.

Proof of (ii). This follows immediately from (i) and the disjointness of the sets underlying the $\Gamma_\alpha$. For if $\beta < \# \Gamma_\alpha$ for some $\alpha$, then $C^{\beta+1}(\Gamma_\alpha) > C^\beta(\Gamma_\alpha) \Rightarrow C^{\beta+1}(T) = \Sigma \{ C^{\beta+1}(\Gamma_\alpha) \mid \alpha \in A \} > \Sigma \{ C^\beta(\Gamma_\alpha) \mid \alpha \in A \} = C^\beta(T)$, which shows $| \Gamma_\alpha | \leq | \alpha |$. On the other hand, $C^{\beta+1}(T) = \Sigma \{ C^{\beta+1}(\Gamma_\alpha) \mid \alpha \in A \} = \Sigma \{ C^\beta(\Gamma_\alpha) \mid \alpha \in A \} = C^\beta(T)$, which shows $| \Gamma_\alpha | = | \alpha |$.  

Lemma 1. For all ordinals $\alpha$ there exists a set of sets $\Gamma_\alpha$ satisfying $| \Gamma_\alpha | = \alpha$. If $\alpha$ is infinite, of cardinality $\aleph_\alpha$, then $\Gamma_\alpha$ can be chosen so that $\bigcup \Gamma_\alpha$ has cardinality $\aleph_\alpha$.

Proof. We induct on the theorem and on the additional property $\bigcup \Gamma_\alpha \subseteq C^\alpha(\Gamma_\alpha)$, but $\forall \beta < \alpha$, $\bigcup \Gamma_\alpha \subseteq C^\beta(\Gamma_\alpha)$. The theorem follows when $\alpha = 0$, trivially and when $\alpha = 1$, letting

$$\Gamma_1 = \{ \{ 1, 2, \ldots, n \} \mid n \in N, \text{ the natural numbers} \},$$

we have that $N \subseteq \Gamma_1$, but $N \subseteq C^1(\Gamma_1) = C^2(\Gamma_1)$.

Assume the theorem and the additional property hold for all ordinals less than $\alpha$, $\alpha > 1$.

Case 1. $\alpha$ is a limit ordinal. For each $\beta < \alpha$ choose $\Gamma_\beta$ satisfying the inductive hypotheses such that the collection $\{ \bigcup \Gamma_\beta \mid \beta < \alpha \}$ of underlying sets is pairwise disjoint. Define $\Gamma_\alpha = \Sigma \{ \Gamma_\beta \mid \beta < \alpha \}$. By (ii) of the proposition we have immediately that $| \Gamma_\alpha | = \alpha$. Now $\bigcup \Gamma_\alpha = \bigcup \{ \bigcup \Gamma_\beta \mid \beta < \alpha \}$, and each $\bigcup \Gamma_\beta \in C^\alpha(\Gamma_\alpha)$, implying by (i) of the proposition $\bigcup \Gamma_\alpha \subseteq \Sigma \{ C^\alpha(\Gamma_\beta) \mid \beta < \alpha \} = C^\alpha(\Gamma_\alpha)$. On the other hand, if $\mu < \alpha$, then $\bigcup \Gamma_{\mu+1} \subseteq \Sigma \{ C^\alpha(\Gamma_\beta) \mid \beta < \alpha \} = C^\alpha(T)$. Thus $\Gamma_\alpha$ satisfies all inductive hypotheses.

Case 2. $\alpha = \gamma + 1$. Let $\Lambda$ be a set of sets satisfying $| \Lambda | = \gamma$ and the other inductive hypotheses. Let $\{ \Lambda_i \mid i = 1, 2, \ldots \}$ be copies of $\Lambda$ obtained by indexing the elements of $\Lambda \Lambda$ with the $i$'s so that the collection $\{ \bigcup \Lambda_i \mid i = 1, 2, \ldots \}$ of underlying sets is pairwise disjoint.

Define

$$\Gamma_\alpha = \{ X \mid X = L_1 \cup L_2 \cup \cdots \cup L_{N-1} \cup (\bigcup \Lambda_N), \text{ where } L_i \in \Lambda_i, N \geq 1 \}.$$
Suppose for all $\mu$ such that $\mu < \beta \leq \gamma$, any $X \in C^\alpha(\Gamma)$ is of the form $S_1 \cup \cdots \cup S_{N-1} \cup (\bigcup A_i)$ where $\forall i = 1, \ldots, N-1$, $S_i \in C^\beta(\Lambda_i)$. Call $N$ the length of $X$. (This is clearly so if $\beta = 1$.) Let $\Delta = \bigcup \{ C^\beta(\Gamma) \mid \mu < \beta \}$. Suppose some member $X$ of $C^\alpha(\Gamma)$ is realized by the upper-directed cover $\{ X_\rho \mid \rho \in R \}$ where $\forall \rho \in R$, $X_\rho \in \Delta$. Further, let $\rho_1, \rho_2 \in R$ be such that $X_{\rho_1} \subseteq X_{\rho_2}$. Say $X_{\rho_1} = S_1 \cup \cdots \cup S_{N-1} \cup (\bigcup A_i)$, $S_i \in C^\alpha(\Lambda_i)$, $\mu_1 < \beta$, and $X_{\rho_2} = T_1 \cup \cdots \cup T_{M-1} \cup (\bigcup A_i)$, $T_i \in C^\alpha(\Lambda_i)$, $\mu_2 < \beta$. If $N 
eq M$, it must be the case that either $S_i \cup A_i$ for some $i$, or $T_i \cup A_i$ for some $i$, which is impossible by the inductive hypotheses on $U\Delta$. Thus $N = M$. Hence all members of the cover $\{ X_\rho \mid \rho \in R \}$ must have the same length $N$, and we may write $X_\rho = S_1 \cup \cdots \cup S_{N-1} \cup (\bigcup A_i)$, where $S_i \in C^\alpha(\Lambda_i)$ for some $\mu < \beta$. The disjointness of the underlying sets of the $\Lambda_i$ now yields that, for $1 \leq i \leq N-1$, $\{ S_i \mid \rho \in R \}$ is an upper-directed cover for $X \cap (\bigcup A_i)$. Thus $X$, an arbitrary element of $C^\alpha(\Lambda)$, is of the form $T_1 \cup \cdots \cup T_{N-1} \cup (\bigcup A_i)$, where $T_i \in C^\beta(\Lambda_i)$, $1 \leq i \leq N-1$.

In particular, the above argument shows that the members of $C^\gamma(\Gamma)$ are of the form

\begin{equation}
S_1 \cup \cdots \cup S_{N-1} \cup (\bigcup A_i), \quad S_i \in C^\gamma(\Lambda_i), \quad 1 \leq i \leq N-1.
\end{equation}

Since $|\Lambda| = \gamma$ and $\bigcup A_i \subseteq C^\gamma(\Lambda)$, it follows that for each $N \geq 1$, $(\bigcup A_i) \cup \cdots \cup (\bigcup A_i) \subseteq C^\gamma(\Gamma)$. Hence $U\Gamma_\alpha = \bigcup \{ \bigcup A_i \mid i = 1, 2, \ldots \}$ has an upper-directed cover of subsets in $C^\gamma(\Gamma)$. But $U\Gamma_\alpha \subseteq C^\gamma(\Gamma)$ since $U\Gamma_\alpha$ is not of the form (*)

It remains to show that $C^{\gamma+1}(\Gamma) = C^{\gamma+2}(\Gamma)$. We will omit the details; however, from the form (*), the following characterization of the members of $C^{\gamma+1}(\Gamma)$ is easily obtained: $X \in C^{\gamma+1}(\Gamma)$ if and only if $X = S_1 \cup \cdots \cup S_{\infty} \subseteq (\bigcup A_i)$ where either (1) each $S_i \in C^\alpha(\Lambda_i)$ and $S_i \subseteq \bigcup A_i$ cofinally in $S_1, \ldots, S_{\infty}$ or (2) $X \in C^\gamma(\Gamma)$ (and hence the $S_i$ are empty after a point).

From this it is easy to see that any directed system of sets in $C^{\gamma+1}(\Gamma)$ again yields a member of $C^{\gamma+1}(\Gamma)$.

Thus $\Gamma_\alpha$ satisfies the inductive hypotheses.

The example given for $|\Gamma| = \omega_1$ at the outset was such that $U\Gamma_\omega$ had cardinality $d$ of the natural numbers. If $\alpha$ is a nonlimit ordinal, $\alpha = \gamma + 1$, and $U\Gamma_\gamma$ is of infinite cardinality $\sigma$, then $\Gamma_\alpha$, as constructed, has cardinality $d \sigma = \sigma$. Thus, as constructed, $U\Gamma_\alpha$ has cardinality $d = \omega$, since at limit ordinals $\alpha$, $U\Gamma_\alpha$ will have cardinality $\sum_{\beta < \alpha} \sigma_\beta$, where $\sigma_\beta$ is the cardinality of $U\Gamma_\beta$.

It is thus clear that for all infinite ordinals $\alpha$, $U\Gamma_\alpha$ will have cardinality $\omega$, and the proof is complete.
Lemma 2. If $A$ is a countably infinite set then there exists an uncountable set of subsets of $A$ such that no containments hold between distinct members.

Proof. Let $\{A_i \mid i = 1, 2, \ldots\}$ be a partition of $A$ such that each $A_i$ is countably infinite. Define $K \subset P(A)$ by $K = \{B \subset A \mid B$ contains exactly one element from each $A_i\}$. The cardinality of $K$ is $d^d = 2^d$, and if $B_1, B_2 \in K$, clearly $B_1 \cap B_2$, unless $B_1 = B_2$.

Corollary. There exists a set of cardinality $c$ of torsion abelian groups, $T = \{T_\alpha \mid \alpha \in K\}$ such that $\forall \alpha_1, \alpha_2 \in K, \alpha_1 \neq \alpha_2 \Rightarrow T_{\alpha_1} \nsubseteq T_{\alpha_2}$.

Proof. Let $A$ be a countably infinite set of primes. Applying Lemma 2, $\exists K \subset P(A)$ such that $K$ is uncountable and for any $B_1, B_2 \in K, B_1 \neq B_2 \Rightarrow B_1$ contains some prime not in $B_2$. For $B \in K$, define $T_B = \sum_{p \in B} J_p$ (the direct sum), where $J_p$ is a cyclic group of order $p$. We claim that the set of groups $\{T_B \mid B \in K\}$ is the desired set. For suppose $B_1, B_2 \in K$ and $B_1 \neq B_2$. Then if $p$ is a prime in $B_1 \setminus B_2$, we have that $T_{B_1}$ has an element of order $p$, whereas $T_{B_2}$ does not. Hence $T_{B_1} \nsubseteq T_{B_2}$.

We will also need several properties of free products of groups.

Lemma 3. Let $\{G_\delta \mid \delta \in \Delta\}$ and $\{H_\lambda \mid \lambda \in \Lambda\}$ be arbitrary collections of groups, and put $R = (\prod_{\delta \in \Delta} G_\delta) \ast (\prod_{\lambda \in \Lambda} H_\lambda)$. Then $(\prod_{\delta \in \Delta} G_\delta) \cap (\prod_{\lambda \in \Lambda} H_\lambda)$ is trivial.

Lemma 4. If $\{G_\delta \mid \delta \in \Delta\}$ is a collection of groups, and $H$ is a freely indecomposable group, but not infinite cyclic, satisfying $H \subset G = \prod_{\delta \in \Delta} G_\delta$, then $\exists \delta \in \Delta$ such that $H$ is a subgroup of a conjugate of $G_\delta$ in $G$.

Proof. Suppose $H \subset G = \prod_{\delta \in \Delta} G$ where $H$ is freely indecomposable but not infinite cyclic. By the subgroup theorem for free products [1, p. 17], $H = F \ast \prod_{v \in V} H_v$ where $F$ is a free group and $\forall v \in V, H_v$ is conjugate in $G$ to a subgroup of $G_\delta$ for some $\delta \in \Delta$. By another theorem [1, p. 26], any two free decompositions of a group possess isomorphic refinements. Hence, since $H$ is freely indecomposable, $F \ast \prod_{v \in V} H_v$ must have exactly one nontrivial factor. $H$ cannot be isomorphic to $F$ since the only freely indecomposable free group is infinite cyclic (or trivial). Thus $H = H_v$ for some $v \in V$. The lemma follows.

We can now prove the desired theorem.

Theorem. For any ordinal $\alpha$ of cardinality $\leq c$, there exists a class of groups $B_\alpha$ such that $L^\alpha(B_\alpha) = L^{\alpha+1}(B_\alpha)$, but $L^\beta(B_\alpha) < L^{\beta+1}(B_\alpha)$ when $\beta < \alpha$. 
Proof. Let $T = \{ T_\alpha \alpha \in K \}$ be the class of torsion abelian groups of the corollary to Lemma 2. By virtue of Lemma 1, $\exists P \subset P(K)$ such that $|P| = \alpha$. If $Y \subset K$, define $F_Y = \prod_{\gamma \in Y} T_\gamma$ and put $F = \{ F_Y \mid Y \subset K \}$.

Define the class $B_a = \{ F_Y \mid Y \in P \}$.

Suppose $\{ F_i \mid i \in \Lambda \}$ is an upper-directed cover of subgroups in $F$ for some $G \in F$, $G = F_Y$. We assert that the set of sets $\{ l \mid l \in \Lambda \}$ is an upper-directed cover of subsets for $Y$, since

1. If $F_i \subset F_{i'}$, then each free factor of $F_{i'}$, by Lemma 4, is isomorphically contained in some free factor of $F_i$, and so by the property of the $\{ T_\alpha \mid \alpha \in K \}$ these free factors are isomorphic. This shows $i_1 \subset i_2$. Hence $\{ l \mid l \in \Lambda \}$ is upper-directed provided $\{ F_i \mid i \in \Lambda \}$ is also.

2. Consider, by Lemma 4, all of the conjugate subgroups of free factors of $G$ to which the free factors of the $\{ F_i \mid i \in \Lambda \}$ belong. If no conjugate of some free factors of $G$ occurs, then Lemma 3 is violated since $\{ F_i \mid i \in \Lambda \}$ covers $G$. Hence all free factors of $G$ are represented, and so $\{ l \mid l \in \Lambda \}$ covers $Y$.

Thus any such group-theoretic covering yields a set-theoretic covering according to the correspondence $T_x \rightarrow x$. Likewise any set-theoretic covering yields a group-theoretic covering.

Considering the local sequence $B_a, L(B_a), \ldots, L^q(B_a), \ldots$ the theorem will be proved if we can eliminate the possibility that, at some stage in the sequence of local covers leading to any $G \in L^q(B_a) \cap F$, some group $H \in F$ occurs. Since the sequence of local covers leading to $G$ is well-ordered, such an $H$ must occur for a first time at some stage. Hence, WLOG, we may assume that $H$ possesses an upper-directed cover $\{ F_\gamma \mid \gamma \in \Gamma \}$ of subgroups in $F$.

Since $H \subset G \subset F$, by the subgroup theorem for free products we have $H \approx Q \ast \prod_{\rho \in R} F_\rho$, where $Q$ is a free group and each $\tau_\rho$ is isomorphic to a subgroup of some member of $T$. Each free factor $T_{\gamma_i}$ of each $F_\gamma$, $\gamma \in \Gamma$, by Lemma 4, is isomorphically contained in some $\tau_\rho$, and hence, for each such $\tau_\rho$, $\tau_\rho \approx T_{\alpha_p}$. Since $\{ F_\gamma \mid \gamma \in \Gamma \}$ covers $H$, by Lemma 3, all of the $\tau_\rho$, $\rho \in R$, are obtained in this way, and consequently $H \approx \prod_{\rho \in R} F_{\alpha_p}$. In order to show $H \subset F$, which will establish the theorem, we must show that no two $T_{\alpha_p}, T_{\alpha_p}$ with $\rho_1, \rho_2 \in R, \rho_1 \neq \rho_2$, are isomorphic. Suppose $T_{\alpha_p} \approx T_{\alpha_p}$. Then by Lemmas 3 and 4 and the covering property of $\{ F_\gamma \mid \gamma \in \Gamma \}$, $\exists F_{\gamma_1}, F_{\gamma_2}, F_{\gamma_3}$ satisfying:

1. Some free factor of $F_{\gamma_1}$ is conjugate to $T_{\alpha_p}$ in $\prod_{\rho \in R} F_{\alpha_p}$.
2. Some free factor of $F_{\gamma_2}$ is conjugate to $T_{\alpha_p}$ in $\prod_{\rho \in R} F_{\alpha_p}$.
3. $F_{\gamma_1} \subset F_{\gamma_2}$ and $F_{\gamma_2} \subset F_{\gamma_3}$.

This implies that $T_{\alpha_p}$ and $T_{\alpha_p}$ are conjugate in $\prod_{\rho \in R} T_{\alpha_p}$ a contradiction since $T_{\alpha_p}$ and $T_{\alpha_p}$ are distinct free factors of $\prod_{\rho \in R} T_{\alpha_p}$.

This completes the proof.
It will be observed that the only group-theoretic property of the "incomparable" set, \( \{ T_\alpha \mid \alpha \in A \} \), of torsion abelian groups used in the proof was the each \( T_\alpha \) was freely indecomposable and not isomorphic to any proper subgroup of itself. Since the set-theoretic lemma was proved for arbitrary ordinals, a stronger result about local sequences of groups will follow whenever a larger "incomparable" set of such freely indecomposable groups can be displayed. The author has not been able to find such a set of cardinality greater than \( \mathfrak{c} \).

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Bibliography


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