EXISTENCE OF REGULAR FINITE INVARIANT MEASURES FOR MARKOV PROCESSES

F. DENNIS SENTILLES

Let $S$ be a locally compact Hausdorff space and let $P(t, x, \cdot)$ be a transition function on the Borel subsets of $S$ with $P(t, x, S) \leq a > 0$ for all $t \geq 0$, $x \in S$. In a recent paper V. E. Beneš [1] obtains several necessary and sufficient conditions that $P(t, x, \cdot)$ have an invariant measure $\mu$ in the space $M(S)^+$ of strictly positive bounded regular Borel measures on $S$ in the presence of several overriding conditions on the function $P$ and the space $S$. We propose to eliminate the need for these conditions by making use of the fact that $M(S)$ is the dual of the locally convex space $C(S)$ of bounded continuous functions on $S$ with the strict topology of Buck [3] (see also [9] for a more general discussion of this topology) and replace the use of weak compactness in $M(S)$ in [1] by that of $\beta$-weak* compactness studied by Conway [4] (these two notions of compactness are studied in [8]) or equivalently, in the notation of [6, p. 32], $\sigma(M(S), C(S))$-compactness.

The referee has pointed out that our results are also an improvement of some of the results in [2], where the same author does replace weak compactness in $M(S)$ by $\sigma(M(S), C_0(S))$-compactness in the presence of certain other conditions. After proving our main theorem we will discuss [2] in more detail.

Our only assumption on $P$ is that for each $t \geq 0$ the function $[T_tf](x) = \int f(s)P(t, x, ds)$ belongs to $C(S)$ for all $f \in C(S)$. Equivalently, the function $x \rightarrow P(t, x, \cdot)$ is a continuous function on $S$ into $M(S)$ with the $\sigma(M(S), C(S)) = \sigma(M, C)$ topology, for in this topology a net $\nu_a \rightarrow \nu$ if and only if $\int f \nu_a = \int f \nu$ for all $f \in C(S)$. It follows from [7] that for each $\mu \in M(S)$ and Borel set $E$ the formula

$$(U_{\mu}E) = \int_S P(t, s, E)\mu(ds)$$

defines a measure in $M(S)$. (An appeal to [7] also yields [2, Lemma 1].) Moreover, $\{ U_t : t \geq 0 \}$ is a semigroup of $\sigma(M, C)$-continuous operators in $M(S)$ and $\langle g, U_{\mu} \rangle = \int_S \int_S \langle s, g \rangle P(t, x, ds)\mu(dx)$ for any bounded Borel measurable function $g$ on $S$.

We denote by $C_0(S)$ those functions $f \in C(S)$ such that

$$\{ s \in S : |f(s)| \geq \varepsilon \}$$

Received by the editors July 25, 1968.
EXISTENCE OF REGULAR FINITE INVARIANT MEASURES 319

is compact in $S$ for each $\epsilon > 0$. A nonnegative Borel measurable function $g$ on $S$ will be called a moment if there exists a sequence $\{K_n\}$ of compact subsets of $S$ with $K_n \subset K_{n+1}^0$ (the interior of $K_{n+1}$) such that \( \lim_{n \to \infty} \sup_{x \in K_n} g(x) = \infty \), with $g$ bounded away from zero on $U_{n-1}^* K_n$. If $g$ is a moment it is easy to see that $\exists \phi \in C_0(S)^+$ such that $1/g(x) \leq \phi(x)$ for $x \in U_{n-1}^* K_n$.

Our main result which includes that of [1] is

**Theorem.** The following are equivalent:

(a) $P(t, x, \cdot)$ has an invariant measure in $M(S)^+$.  
(b) $\exists \mu \in M(S)^+$ and a moment $g$ such that for $t \geq 0$ $U_t \mu(S \setminus U_{n-1}^* K_n) = 0$ and $\sup_{t \geq 0} (g, U_t \mu) < \infty$.  
(c) $\exists \mu \in M(S)^+$, $\phi \in C_0(S)^+$ such that all measures $U_t \mu$ vanish off the zeroes of $\phi$ and $\sup_{t \geq 0} (1/\phi, U_t \mu) < \infty$.  
(d) $\exists \mu \in M(S)^+$, $K$ a convex $\sigma(M, C)$-compact subset of $M(S)$ such that $\{U_t \mu : t \geq 0\} \subset K$.  
(e) $\exists \mu \in M(S)^+$ such that the measures $\{U_t \mu : t \geq 0\}$ are uniformly countably additive.  
(f) $\exists \mu \in M(S)^+$, $K$ a $\sigma(M(S), M(S)^*)$ compact subset of $M(S)$ such that $\{U_t \mu : t \geq 0\} \subset K$.

**Proof.** We will show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and that (a) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (d).  

(a) $\Rightarrow$ (b). If $\mu$ is the assumed regular invariant measure on the locally compact space $S$, then $\exists$ a sequence $\{K_n\}$ of compacta such that $K_n \subset K_{n+1}^0$ and $\mu(S \setminus K_n) < 1/n^3$. Setting $g(x) = n$ for $x \in K_n$ and $g(x) = 1$ for $x \in U_{n-1}^* K_n$ yields the proof.

(b) $\Rightarrow$ (c). We choose $\phi \in C_0(S)^+$ so that $1/g \leq \phi$ on $U_{n-1}^* K_n$. Then $\{x : \phi(x) = 0\} \subset S \setminus U_{n-1}^* K_n$ and $(1/\phi, U_t \mu) \leq (g, U_t \mu)$ for all $t \geq 0$.

(c) $\Rightarrow$ (d). By [4, Theorem 2.1] $\{U_t \mu : t \geq 0\}$ is an equicontinuous subset of $M(S)$ as the dual of $C(S)$ with the strict topology. Hence by [6, p. 61, Theorem 6 and p. 46], $\{U_t \mu : t \geq 0\}$ is a subset of some $\sigma(M, C)$ convex compact set in $M(S)$.

(d) $\Rightarrow$ (a) The proof is essentially like that of [1]. Note that the $\sigma(M, C)$-closure $Q$ of the convex hull of $\{U_t \mu : t \geq 0\}$ is $\sigma(M, C)$-compact, assigns a measure of at least the number $a \mu(S) > 0$ to $S$, and is invariant under all operators $U_t$, $t > 0$. By the Markov-Kakutani Theorem [4, p. 456], $\exists \nu \in Q$ such that $U_t \nu = \nu$ for all $t \geq 0$ proving (a).

It is clear that (a) $\Rightarrow$ (e) while (e) $\Rightarrow$ (f) by [5, pp. 341 and 430]. If (f) holds, then by [5, p. 434] the (variation) norm closure of the convex hull of $K$ is $\sigma(M(S), M(S)^*)$-compact and hence also $\sigma(M, C)$-compact. Consequently (d) holds.

Before considering the results in [2] we note that in the light of [4, Theorems 2.1 and 2.2] the above statements are equivalent to:
(g) \( \exists \mu \in M(S)^+ \) such that \( \{ U_t \mu : t \geq 0 \} \) is tight. That is, given \( \epsilon > 0 \), there exists a compact \( K \subseteq S \) such that

\[
U_t \mu(S \setminus K) < \epsilon \quad \text{for all } t \geq 0,
\]

while, as noted in [8, Section II and Theorem 2] condition (f) is equivalent to

(h) \( \exists \mu \in M(S)^+ \) such that for each open set \( U \subseteq S \) and each \( \epsilon > 0 \), there exists a compact set \( K \subset U \) such that

\[
U_t \mu(U \setminus K) < \epsilon \quad \text{for all } t \geq 0.
\]

Finally, Conway has noted in his dissertation that (g) holds if and only if \( \exists \mu \in M(S)^+ \) such that every sequence in \( \{ U_t \mu : t \geq 0 \} \) has a \( \sigma(M, C) \) cluster point in \( M(S) \) since we are here dealing only with positive measures.

In closing we first note that (g) is similar to condition (iii) in [2]. Secondly, when the integrals mentioned below exist in \( M(S) \), we obtain

(k) \( \exists \mu \in M(S)^+ \) and \( K \) a \( \sigma(M, C) \)-compact subset of \( M(S) \) such that

\[
1/t_n \int_0^{t_n} U_s \mu ds \subseteq K \quad \text{for some sequence } t_n \to \infty\text{ as an analogue of [2, (iv)] readily from (d).}
\]

One can then use (k) to imply (a), for the sequence \( \{ 1/t_n \int_0^{t_n} U_s \mu ds \} \) must then have a nonzero \( \sigma(M, C) \) cluster point in \( M(S)^+ \) which, as in the latter part of the proof of (v)\( \Rightarrow \) (i) in [2], is an invariant measure for \( \{ U_t : t > 0 \} \). Thus, as the referee suggested was possible, our main theorem can be obtained as in [2] without the use of the Markov-Kakutani theorem.

References


University of Missouri