EXISTENCE OF REGULAR FINITE INVARIANT MEASURES FOR MARKOV PROCESSES

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Let S be a locally compact Hausdorff space and let $P(t, x, \cdot)$ be a transition function on the Borel subsets of S with $P(t, x, S) \leqq a > 0$ for all $t \geqq 0$, $x \in S$. In a recent paper V. E. Beneš [1] obtains several necessary and sufficient conditions that $P(t, x, \cdot)$ have an invariant measure μ in the space $M(S)^+$ of strictly positive bounded regular Borel measures on S in the presence of several overriding conditions on the function P and the space S. We propose to eliminate the need for these conditions by making use of the fact that $M(S)$ is the dual of the locally convex space $C(S)$ of bounded continuous functions on S with the strict topology of Buck [3] (see also [9] for a more general discussion of this topology) and replace the use of weak compactness in $M(S)$ in [1] by that of β-weak* compactness studied by Conway [4] (these two notions of compactness are studied in [8]) or equivalently, in the notation of [6, p. 32], $\sigma(M(S), C(S))$-compactness.

The referee has pointed out that our results are also an improvement of some of the results in [2], where the same author does replace weak compactness in $M(S)$ by $\sigma(M(S), C_0(S))$-compactness in the presence of certain other conditions. After proving our main theorem we will discuss [2] in more detail.

Our only assumption on P is that for each $t \geqq 0$ the function $[T_t f](x) = \int_S f(s) P(t, x, ds)$ belongs to $C(S)$ for all $f \in C(S)$. Equivalently, the function $x \rightarrow P(t, x, \cdot)$ is a continuous function on S into $M(S)$ with the $\sigma(M(S), C(S)) = \sigma(M, C)$ topology, for in this topology a net $\nu_\alpha \rightarrow \nu$ if and only if $\int_S f d\nu_\alpha \rightarrow \int_S f d\nu = \langle f, \nu \rangle$ for all $f \in C(S)$. It follows from [7] that for each $\mu \in M(S)$ and Borel set E the formula

$$(U_t \mu)(E) = \int_S P(t, s, E) \mu(ds)$$

defines a measure in $M(S)$. (An appeal to [7] also yields [2, Lemma 1].) Moreover, $\{U_t : t \geqq 0\}$ is a semigroup of $\sigma(M, C)$-continuous operators in $M(S)$ and $\langle g, U_t \mu \rangle = \int_S \int_S g(s) P(t, x, ds) \mu(dx)$ for any bounded Borel measurable function g on S.

We denote by $C_0(S)$ those functions $f \in C(S)$ such that

$$\{s \in S : |f(s)| \geqq \epsilon\}$$

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is compact in S for each $\epsilon>0$. A nonnegative Borel measurable function g on S will be called a moment if $\exists$ a sequence $\{K_n\}$ of compact subsets of S with $K_n \subset K_{n+1}^0$ (the interior of K_{n+1}) such that $\lim_{n\to\infty} \sup_{x\in K_n\setminus K_{n+1}^0} g(x)=\infty$, with g bounded away from zero on $\bigcup_{n=1}^{\infty} K_n$. If g is a moment it is easy to see that $\exists \phi \in C_0(S)^+$ such that $1/g(x) \leqq \phi(x)$ for $x \in \bigcup_{n=1}^{\infty} K_n$.

Our main result which includes that of [1] is

THEOREM. *The following are equivalent:*

(a) *$P(t, x, \cdot)$ has an invariant measure in $M(S)^+$.*

(b) *$\exists \mu \in M(S)^+$ and a moment g such that for $t \geqq 0$ $U_t\mu(S\setminus\bigcup_{n=1}^{\infty} K_n)$ $=0$ and $\sup_{t\geqq 0}\langle g, U_t\mu\rangle<\infty$.*

(c) *$\exists \mu \in M(S)^+$, $\phi \in C_0(S)^+$ such that all measures $U_t\mu$ vanish off the nonzeroes of ϕ and $\sup_{t\geqq 0}\langle 1/\phi, U_t\mu\rangle<\infty$.*

(d) *$\exists \mu \in M(S)^+$, K a convex $\sigma(M, C)$-compact subset of $M(S)$ such that $\{U_t\mu: t\geqq 0\}\subset K$.*

(e) *$\exists \mu \in M(S)^+$ such that the measures $\{U_t\mu: t\geqq 0\}$ are uniformly countably additive.*

(f) *$\exists \mu \in M(S)^+$, K a $\sigma(M(S), M(S)^*)$ compact subset of $M(S)$ such that $\{U_t\mu: t\geqq 0\}\subset K$.*

PROOF. We will show that (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(a) and that (a)$\Rightarrow$(e)$\Rightarrow$(f)$\Rightarrow$(d).

(a)$\Rightarrow$(b). If μ is the assumed regular invariant measure on the locally compact space S, then $\exists$ a sequence $\{K_n\}$ of compacta such that $K_n\subset K_{n+1}^0$ and $\mu(S\setminus K_n)<1/n^3$. Setting $g(x)=n$ for $x\in K_n$ and $g(x)=1$ for $x\notin \bigcup_{n=1}^{\infty} K_n$ yields the proof.

(b)$\Rightarrow$(c). We choose $\phi\in C_0(S)^+$ so that $1/g\leqq\phi$ on $\bigcup_{n=1}^{\infty} K_n$. Then $\{x: \phi(x)=0\}\subset S\setminus\bigcup_{n=1}^{\infty} K_n$ and $\langle 1/\phi, U_t\mu\rangle \leqq \langle g, U_t\mu\rangle$ for all $t\geqq 0$.

(c)$\Rightarrow$(d). By [4, Theorem 2.1] $\{U_t\mu: t\geqq 0\}$ is an equicontinuous subset of $M(S)$ as the dual of $C(S)$ with the strict topology. Hence by [6, p. 61, Theorem 6 and p. 46], $\{U_t\mu: t\geqq 0\}$ is a subset of some $\sigma(M, C)$ convex compact set in $M(S)$.

(d)$\Rightarrow$(a) The proof is essentially like that of [1]. Note that the $\sigma(M, C)$-closure Q of the convex hull of $\{U_t\mu: t\geqq 0\}$ is $\sigma(M, C)$-compact, assigns a measure of at least the number $a\mu(S)>0$ to S, and is invariant under all operators U_t, $t>0$. By the Markov-Kakutani Theorem [4, p. 456], $\exists \nu\in Q$ such that $U_t\nu=\nu$ for all $t\geqq 0$ proving (a).

It is clear that (a)$\Rightarrow$(e) while (e)$\Rightarrow$(f) by [5, pp. 341 and 430]. If (*f*) holds, then by [5, p. 434] the (variation) norm closure of the convex hull of K is $\sigma(M(S), M(S)^*)$-compact and hence also $\sigma(M, C)$-compact. Consequently (d) holds.

Before considering the results in [2] we note that in the light of [4, Theorems 2.1 and 2.2] the above statements are equivalent to:

(g) $\exists \mu \in M(S)^+$ such that $\{U_t\mu: t \geqq 0\}$ is tight. That is, given $\epsilon > 0 \exists$ a compact $K \subset S$ such that

$$U_t\mu(S \backslash K) < \epsilon \text{ for all } t \geqq 0,$$

while, as noted in [8, Section II and Theorem 2] condition (f) is equivalent to

(h) $\exists \mu \in M(S)^+$ such that for each open set $U \subset S$ and each $\epsilon > 0 \exists$ a compact set $K \subset U$ such that $U_t\mu(U/K) < \epsilon$ for all $t \geqq 0$.

Finally, Conway has noted in his dissertation that (g) holds if and only if $\exists \mu \in M(S)^+$ such that every sequence in $\{U_t\mu: t \geqq 0\}$ has a $\sigma(M, C)$ cluster point in $M(S)$ since we are here dealing only with positive measures.

In closing we first note that (g) is similar to condition (iii) in [2]. Secondly, when the integrals mentioned below exist in $M(S)$, we obtain

(k) $\exists \mu \in M(S)^+$ and K a $\sigma(M, C)$-compact subset of $M(S)$ such that $1/t_n \int_0^{t_n} U_s\mu ds \in K$ for some sequence $t_n \rightarrow \infty$ as an analogue of [2, (iv)] readily from (d). One can then use (k) to imply (a), for the sequence $\{1/t_n \int_0^{t_n} U_s\mu ds\}$ must then have a nonzero $\sigma(M, C)$ cluster point in $M(S)^+$ which, as in the latter part of the proof of (v)$\Rightarrow$(i) in [2], is an invariant measure for $\{U_t: t > 0\}$. Thus, as the referee suggested was possible, our main theorem can be obtained as in [2] without the use of the Markov-Kakutani theorem.

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