EXISTENCE OF REGULAR FINITE INVARIANT MEASURES FOR MARKOV PROCESSES

F. DENNIS SENTILLES

Let $S$ be a locally compact Hausdorff space and let $P(t, x, \cdot)$ be a transition function on the Borel subsets of $S$ with $P(t, x, S) \leq a > 0$ for all $t \geq 0$, $x \in S$. In a recent paper V. E. Beneš [1] obtains several necessary and sufficient conditions that $P(t, x, \cdot)$ have an invariant measure $\mu$ in the space $M(S)^+$ of strictly positive bounded regular Borel measures on $S$ in the presence of several overriding conditions on the function $P$ and the space $S$. We propose to eliminate the need for these conditions by making use of the fact that $M(S)$ is the dual of the locally convex space $C(S)$ of bounded continuous functions on $S$ with the strict topology of Buck [3] (see also [9] for a more general discussion of this topology) and replace the use of weak compactness in $M(S)$ in [1] by that of $\beta$-weak* compactness studied by Conway [4] (these two notions of compactness are studied in [8]) or equivalently, in the notation of [6, p. 32], $\sigma(M(S), C(S))$-compactness.

The referee has pointed out that our results are also an improvement of some of the results in [2], where the same author does replace weak compactness in $M(S)$ by $\sigma(M(S), C_0(S))$-compactness in the presence of certain other conditions. After proving our main theorem we will discuss [2] in more detail.

Our only assumption on $P$ is that for each $t \geq 0$ the function $[T_\mu f](x) = \int_S f(s)P(t, x, ds)$ belongs to $C(S)$ for all $f \in C(S)$. Equivalently, the function $x \mapsto P(t, x, \cdot)$ is a continuous function on $S$ into $M(S)$ with the $\sigma(M(S), C(S)) = \sigma(M, C)$ topology, for in this topology a net $\nu_{\sigma} \rightarrow \nu$ if and only if $\int_S f d\nu_{\sigma} \rightarrow \int_S f d\nu = \langle f, \nu \rangle$ for all $f \in C(S)$. It follows from [7] that for each $\mu \in M(S)$ and Borel set $E$ the formula

$$(U_\mu)(E) = \int_S P(t, s, E)\mu(ds)$$

defines a measure in $M(S)$. (An appeal to [7] also yields [2, Lemma 1].) Moreover, $\{U_t: t \geq 0\}$ is a semigroup of $\sigma(M, C)$-continuous operators in $M(S)$ and $\langle g, U_\mu \rangle = \int_S \int_S g(s)P(t, x, ds)\mu(dx)$ for any bounded Borel measurable function $g$ on $S$.

We denote by $C_0(S)$ those functions $f \in C(S)$ such that

$$\{s \in S: |f(s)| \geq \epsilon\}$$

Received by the editors July 25, 1968.
is compact in $S$ for each $\epsilon > 0$. A nonnegative Borel measurable function $g$ on $S$ will be called a moment if there exists a sequence $\{K_n\}$ of compact subsets of $S$ with $K_n \subseteq K_{n+1}$ (the interior of $K_{n+1}$) such that $\lim_{n \to \infty} \sup_{x \in K_n} g(x) = \infty$, with $g$ bounded away from zero on $U_{n-1}^* K_n$. If $g$ is a moment it is easy to see that $\exists \phi \in C_0(S)^+$ such that $1/g(x) \leq \phi(x)$ for $x \in U_{n-1}^* K_n$.

Our main result which includes that of [1] is

**Theorem.** The following are equivalent:

(a) $P(t, x, \cdot)$ has an invariant measure in $M(S)^+$. 
(b) $\exists \mu \in M(S)^+$ and a moment $g$ such that for $t \geq 0$, $U_t(S \setminus U_{n-1}^* K_n) = 0$ and $\sup_{t \geq 0} (g, U_\mu) < \infty$.
(c) $\exists \mu \in M(S)^+$, $\phi \in C_0(S)^+$ such that all measures $U_\mu$ vanish off the zeroes of $\phi$ and $\sup_{t \geq 0} (1/\phi, U_\mu) < \infty$.
(d) $\exists \mu \in M(S)^+$, $K$ a convex $\sigma(M, C)$-compact subset of $M(S)$ such that $\{U_\mu : t \geq 0\} \subseteq K$.
(e) $\exists \mu \in M(S)^+$ such that the measures $\{U_\mu : t \geq 0\}$ are uniformly countably additive.
(f) $\exists \mu \in M(S)^+$, $K$ a $\sigma(M(S), M(S)^*)$ compact subset of $M(S)$ such that $\{U_\mu : t \geq 0\} \subseteq K$.

**Proof.** We will show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and that (a) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (d).

(a) $\Rightarrow$ (b). If $\mu$ is the assumed regular invariant measure on the locally compact space $S$, then there is a sequence $\{K_n\}$ of compacta such that $K_n \subseteq K_{n+1}$ and $\mu(S \setminus K_n) < 1/n^3$. Setting $g(x) = n$ for $x \in K_n$ and $g(x) = 1$ for $x \in U_{n-1}^* K_n$ yields the proof.

(b) $\Rightarrow$ (c). We choose $\phi \in C_0(S)^+$ so that $1/g \leq \phi$ on $U_{n-1}^* K_n$. Then $\{x : \phi(x) = 0\} \subseteq S \setminus U_{n-1}^* K_n$ and $1/\phi, U_\mu \leq (g, U_\mu)$ for all $t \geq 0$.

(c) $\Rightarrow$ (d). By [4, Theorem 2.1], $\{U_\mu : t \geq 0\}$ is an equicontinuous subset of $M(S)$ as the dual of $C(S)$ with the strict topology. Hence by [6, p. 61, Theorem 6 and p. 46], $\{U_\mu : t \geq 0\}$ is a subset of some $\sigma(M, C)$ convex compact set in $M(S)$.

(d) $\Rightarrow$ (a) The proof is essentially like that of [1]. Note that the $\sigma(M, C)$-closure $Q$ of the convex hull of $\{U_\mu : t \geq 0\}$ is $\sigma(M, C)$-compact, assigns a measure of at least the number $a_\mu(S) > 0$ to $S$, and is invariant under all operators $U_t, t > 0$. By the Markov-Kakutani Theorem [4, p. 456], $\exists \nu \in Q$ such that $U_t \nu = \nu$ for all $t \geq 0$ proving (a).

It is clear that (a) $\Rightarrow$ (e) while (e) $\Rightarrow$ (f) by [5, pp. 341 and 430]. If (f) holds, then by [5, p. 434] the (variation) norm closure of the convex hull of $K$ is $\sigma(M(S), M(S)^*)$-compact and hence also $\sigma(M, C)$-compact. Consequently (d) holds.

Before considering the results in [2] we note that in the light of [4, Theorems 2.1 and 2.2] the above statements are equivalent to:
(g) \(\exists \mu \in M(S)^+\) such that \(\{U_t \mu : t \geq 0\}\) is tight. That is, given \(\epsilon > 0\) a compact \(K \subseteq S\) such that
\[
U_t \mu(S \setminus K) < \epsilon \text{ for all } t \geq 0,
\]
while, as noted in [8, Section II and Theorem 2] condition (f) is equivalent to

(h) \(\exists \mu \in M(S)^+\) such that for each open set \(U \subseteq S\) and each \(\epsilon > 0\) a compact set \(K \subseteq U\) such that \(U \mu(U/K) < \epsilon\) for all \(t \geq 0\).

Finally, Conway has noted in his dissertation that (g) holds if and only if \(\exists \mu \in M(S)^+\) such that every sequence in \(\{U_t \mu : t \geq 0\}\) has a \(\sigma(M, C)\) cluster point in \(M(S)\) since we are here dealing only with positive measures.

In closing we first note that (g) is similar to condition (iii) in [2]. Secondly, when the integrals mentioned below exist in \(M(S)\), we obtain

(k) \(\exists \mu \in M(S)^+\) and \(K\) a \(\sigma(M, C)\)-compact subset of \(M(S)\) such that \(1/t_n \int_0^t U_t \mu ds \in K\) for some sequence \(t_n \to \infty\) as an analogue of [2, (iv)] readily from (d). One can then use (k) to imply (a), for the sequence \(1/t_n \int_0^t U_t \mu ds\) must then have a nonzero \(\sigma(M, C)\) cluster point in \(M(S)^+\) which, as in the latter part of the proof of (v) \(\Rightarrow\) (i) in [2], is an invariant measure for \(\{U_t : t > 0\}\). Thus, as the referee suggested was possible, our main theorem can be obtained as in [2] without the use of the Markov-Kakutani theorem.

REFERENCES


UNIVERSITY OF MISSOURI