SOME LARGE O NONLINEAR TAUBERIAN THEOREMS

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The following "large-O" nonlinear Tauberian Theorems are extensions of the "small-O" theorems of Boas [1], Karamata [2] and Pollard [3]. Motivation for these theorems comes from the problem of n-bodies in celestial mechanics. Here bounds on the behavior of the self-potential, \( U \), are sought starting from the knowledge that \( \int U(s)ds = O(t^a) \quad (t \to 0+, \text{ or } t \to \infty) \) and \( |U| \leq C t^a \). Here \( C \) and \( a \) are constants.

In the following the symbols \( f, g \) and \( h \) represent functions which are of the class \( C^2 \) on \((0, \infty)\) and \( \omega, \phi \) represent positive continuous functions.

\( f(t) = O(t) \) has its standard meaning; \( f(t)/t \) is bounded above (but not necessarily below) as \( t \) approaches its specified limit.

**Theorem 1.** If

\[ g(t) = O(t) \quad \text{as } t \to 0+, \quad \text{and} \quad g''(t) \leq \omega(g'(t))O(\phi(t)), \]

where \( \phi(t) \) is integrable, then

\[ g'(t) = O(1) \quad \text{as } t \to 0+. \]

Note that this includes \( \phi(t) = t^a \) where \( a > -1 \).

**Proof.** Since \( g(t) = O(t) \), a \( B > 0 \) can be found such that for all positive \( t \) less than some value, \( |g(t)| \leq Bt \). Define \( A = \{t: |g'(t)| \leq B\} \).

Claim. \( A \) is nonempty and has zero as a limit point. If this were not so then there would exist a neighborhood of zero \((0, t)\) such that \( |g'(t)| > B \). The continuity of \( g' \) implies that it has one sign in this neighborhood. Thus \( |g(t)| = \int_0^t g'(u)du > Bt \), a contradiction.

To simplify the formulas, set \( y = g'(t) \). Then \( dy/dt = g''(t) \) and the hypothesis implies that for positive \( t \) less than some value

\[ dy/dt \leq C \omega(y) \phi(t). \]

As \( \Phi(t) = \int_0^t \phi(u)du \) is a positive increasing function, it follows that

\[ (1) \quad W(y_1) - W(y) = \int_y^{y_1} \frac{dy}{\omega(y)} \leq C \int_t^{y_1} \phi(s)ds < C \Phi(t) \]

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where $W$ is defined by (1) and $t_2 \geq t_1$.

Note that $W$ is an increasing function ($\omega(y) > 0$).

**Claim.** $y$ is eventually bounded above by $2B$. Assume this to be false. Then an $\epsilon > 0$ can be determined such that $W(2B) - W(B) > \epsilon$. Choose $t_2$ small enough so that $C\Phi(t_2) < \epsilon$. Now assume $t_1$ is such that $y_1 \geq 2B$. By properties of set $A$ and the choice of $y_1$, there is a $t$ such that $0 < t < t_1$ and $y(t) = B$. Substituting the above into (1) yields $\epsilon > C\Phi(t_2) > W(y_1) - W(B) \geq W(2B) - W(B) > \epsilon$. From this contradiction it follows that $y$ is bounded above. A similar proof is employed to show that $y$ is bounded below and the proof is completed.

It is clear that the corresponding theorem for $t \to \infty$ holds also. The proof is given most simply by modifying the reasoning rather than a change in variable.

**Theorem 2.** If

$$g(t) = O(t) \quad \text{as} \quad t \to \infty,$$

and

$$g''(t) \leq \omega(g'(t))O(\phi(t)),$$

where $\int_{t}^{\infty} \phi(s) \, ds$ is a convergent integral, then

$$g'(t) = O(1) \quad \text{as} \quad t \to \infty.$$

The above two theorems can be sharpened to include the case $\phi(t) = t^{-1}$. This will be stated and proved here only when $t \to \infty$. The corresponding statement for $t \to 0^+$ is the same.

**Theorem 3.** If $|g(t)| \leq C t$, $t \to \infty$, and $g''(t) \leq \omega(g'(t))t^{-1}$, then $g'$ is bounded above if either:

1. $\int_{t}^{\infty} du/\omega(u)$ is unbounded as $x \to \infty$,
2. $\lim (x^2 - 1) \left[ 1/\omega(Cx) - n/\omega(nCx) \right] > 2/C$ as $x \to \infty$, and $n$ is some constant $> 1$. $g'$ is bounded below if either (1) or (2) is true for $x \to -\infty$.

**Proof.** It will be shown here that $g'$ is bounded above. The proof that it is bounded below is obtained in a similar manner.

**Claim.** If $g' \geq MC$, where $M > 1$, it is so at most on an interval $(t, \lambda t)$ where $\lambda \leq (M + 1)/(M - 1)$.

By the hypothesis and the assumption on $g'$,

$$(\lambda + 1)Ct \geq g(\lambda t) - g(t) = \int_{t}^{\lambda t} g'(s) \, ds \geq MCt(\lambda - 1),$$

or $\lambda \leq (M + 1)/(M - 1)$.

For notational purposes set $y = g'(t)$. Assign a fixed value to $M$ and consider an interval $[t_1, t]$ on which $y \geq MC$ and $y(t_1) = MC$. (If no such interval exists, then $y$ is bounded above by $MC$.) By the conditions imposed upon $t/t_1 = \lambda$ and the second derivative of $g,$
If \( W \) is an unbounded (increasing by virtue of \( \omega > 0 \)) function, then it follows immediately from (2) that \( y \) is bounded above.

If \( W \) is bounded above, a slight modification in terms of extra conditions is necessary to apply the above technique. A sufficient condition would be if, for some \( n>1 \), an \( M < \infty \) could be found such that \( W(nCM) - W(CM) > \ln((M+1/M-1)) \). \( y \) would then be bounded above by \( nCM \).

If the limit exists, such an \( M \) can be found if for some \( n>1 \),

\[
\lim_{x \to \infty} \frac{W(nCx) - W(Cx)}{\ln(x+1)/(x-1)} > 1
\]

or by L'Hospital's rule, if

\[
\lim_{x \to \infty} (x^2 - 1)/(\omega(Cx) - n/\omega(nCx)) > 2/C
\]

Hence under the above conditions, \( g' \) is bounded above.

If \( \omega \) is bounded, then we have as a special case of Theorem 3 a well-known linear Tauberian Theorem: \( g(t) = O(t^\alpha) \), \( g'' = O(t^{-\alpha}) \) as \( t \to \infty \) (or \( t \to 0+ \)).

The above theorems can be rewritten in a form which includes the motivating problem at the beginning of this note.

**Theorem 4.** If

\[
h(t) = O(t^\alpha), \quad t \to \infty, \quad \alpha \neq 0,
\]

\[
|h''(t)| \leq B \cdot |h'(t)|^4 \eta,
\]

and \( h'(t) \) is of one sign after some value of \( t \), then \( h'(t) = O(t^{\alpha-1}) \) if any of the following hold.

(i) \( \eta = (\delta - 1)(\alpha - 1) + \beta + 1 < 0 \).

(ii) \( \delta < 2 \) and \( \eta \leq 0 \).

(iii) \( |h(t)| \leq C t^{\delta}, \ \delta = 2, \ 2BC < \alpha^2 \) and \( \eta \leq 0 \).

**Proof.** It can be assumed without loss of generality that \( h' > 0 \). Define

\[
g(t^\alpha) = h(t) = O(t^\alpha), \quad \alpha g'(t^\alpha) t^{\alpha-1} = h'(t)
\]

and

\[
g''(t^\alpha) \alpha^2 t^{2\alpha-2} + g'(t^\alpha) \alpha(\alpha - 1) t^{\alpha-2} = h''(t).
\]

Note that \( g'(x) \) is of one sign. If \( \alpha = 1 \), the above is reduced to the conditions of Theorems 2 and 3 and this theorem follows.
\( \alpha > 1 \). Substituting the above for \( h \) in the second derivative condition and noting that the term containing \( g' \) is nonnegative, it follows that

\[
\alpha^2 \epsilon^2 g''(\epsilon^2) \leq B \left| g'(\epsilon^2) \right| \delta \left( \frac{1}{\alpha-1} \right)^{\alpha+1} + 1.
\]

By substituting \( x = \epsilon^2 \) and applying the conditions of Theorems 2 and 3, this theorem follows.

If \( 0 < \alpha < 1 \) the proof is the same except the term involving \( g' \) is now negative, so the other inequality available from the second derivative condition is employed. For negative \( \alpha \), \( x \to 0 \) and Theorems 1 and 3 are used. Notice that the absolute value of \( h'' \) is not necessary, only the appropriate inequality depending on the value of \( \alpha \) and the sign of \( h'(t) \).

If \( \delta = 0 \), Theorem 4 has as a special case a well-known linear Tauberian Theorem. Note the interesting returns if \( \delta = 1 \). All that is necessary for all \( \alpha \neq 0 \) is \( \beta \leq -1 \).

Following the reasoning of Pollard [3] Theorem 4 can be extended to the case \( \alpha = \infty \) in the following form.

**Theorem 5.** If

\[
h(t) = O(e^{t}), \quad t \to \infty,
\]

and

\[
|h''(t)| \leq B \left| h'(t) \right| ^{\delta}, \quad \delta < 2,
\]

then

\[
h'(t) = O(e^{t}), \quad t \to \infty.
\]

**Proof.** The function \( g(e^{t}) = h(t) \) satisfies the conditions of Theorem 3 where \( \omega(y) = |y|^{1} + |y| + 1 \).

**References**


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