PRODUCT SPACES FOR WHICH THE STONE-WEIERSTRASS THEOREM HOLDS

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1. **Introduction.** A topological space X is said to be a completely Hausdorff space (or a Stone space) provided that $C(X)$, the set of bounded continuous real valued functions defined on X, is point separating. In the following we call a completely Hausdorff space X an SW space if every point separating subalgebra of $C(X)$ which contains the constants is uniformly dense in $C(X)$. Equivalently, according to Theorem 1, one can define a topological space to be an SW space provided that it can be obtained from a compact Hausdorff space by refining the topology without adding any continuous real valued functions.

In [5] the author proved that if X and Y are SW spaces, one of which is compact, then $X \times Y$ is an SW space. In this note a proof is given that if X and Y are SW spaces, then $X \times Y$ is an SW space if and only if pr_1 is Z-closed, i.e. for every zero set Z of $X \times Y$, $\mathrm{pr}_1(Z)$ is a closed subset of X. Several consequences of this theorem are considered, and some examples are given of noncompact spaces to which it applies.

We use the same terminology as that in [2] or [3]. Given a completely Hausdorff space X, we shall denote by wX the completely regular space which has the same points and the same continuous real valued functions as those of X. $L(X)$ will denote the set of all functions in $C(X)$ which map X into $[0, 1]$.

A filter base on a space X is called an open filter base if the sets belonging to it are open subsets of X. An open filter base $\mathfrak{F}$ is said to be completely regular if for each set $F \in \mathfrak{F}$ there exist a set $G \in \mathfrak{F}$ and a function $f \in L(X)$ such that f vanishes on G and equals 1 on $X - F$. $\mathfrak{F}$ is said to be fixed if $\bigcap \mathfrak{F} \neq \varnothing$.

Our proofs are all based on the following characterization theorem [1].

THEOREM 1 (BANASCHEWSKI). *Let X be a completely Hausdorff space. The following are equivalent.*

(i) *X is an SW space.*

(ii) *wX is compact.*

(iii) *Every completely regular filter base on X is fixed.*

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2. A characterization theorem for products.

LEMMA 2. *Let $\phi: X \to Y$ be a Z-closed open mapping from the space X onto the space Y. Then, for any $f \in L(X)$, the function g, defined by $g(y) = \inf\{f(x) \mid \phi(x) = y\}$, belongs to $L(Y)$.*

PROOF. One readily checks that for any $s < t$,

$$g^{-1}(s, t) = \bigcup_{\lambda > s} Y - \phi(f^{-1}[0, \lambda]) \cap \phi(f^{-1}[0, t)),$$

and it follows from the given properties of ϕ that this set is open.

LEMMA 3. *If X is a completely Hausdorff space, Y an SW space, and $\phi: X \to Y$ a Z-closed open surjection such that each $\phi^{-1}\{y\}$, $y \in Y$, is an SW space, then X is also an SW space.*

PROOF. Let $\mathfrak{F}$ be a completely regular filter base on X. Using Lemma 2, one can show that $\phi(\mathfrak{F})$ is a completely regular filter base on Y. Then there exists $a \in \bigcap \phi(\mathfrak{F})$ and for $A = \phi^{-1}\{a\}$ a point $b \in A \cap (\bigcap \mathfrak{F})$, the latter since the restriction of $\mathfrak{F}$ to A is a completely regular filter base on A.

THEOREM 4. *Let X and Y be SW spaces. The following are equivalent.*
(i) pr_1 *is Z-closed.*
(ii) $X \times Y$ *is an SW space.*
(iii) $w(X \times Y) = wX \times wY$.

PROOF. (i) implies (ii). Since $\{f \circ \mathrm{pr}_i \mid i = 1$ and $f \in C(X)$, or $i = 2$ and $f \in C(Y)\}$ is a point separating subset of $C(X \times Y)$, $C(X \times Y)$ is point separating and $X \times Y$ is a completely Hausdorff space. By Lemma 3, $X \times Y$ is an SW space.

(ii) implies (iii). If $X \times Y$ is an SW space, then $w(X \times Y)$ is compact by Theorem 1. Since $wX \times wY$ is a Hausdorff space and the identity mapping $i: w(X \times Y) \to wX \times wY$ is a continuous bijection, i is a homeomorphism.

(iii) implies (i). (iii) implies that $C(X \times Y) = C(w(X \times Y)) = C(wX \times wY)$. Thus $X \times Y$ and $wX \times wY$ have the same zero sets. Since $\mathrm{pr}_1: wX \times wY \to wX$ is a closed mapping, and the closed subsets of wX are closed subsets of X, (i) holds.

REMARK 5. (i) of Theorem 4 holds if and only if pr_1 is zero set preserving. More generally, one can use Lemma 2 to prove that if $\phi: X \to Y$ is an open surjection for which all $\phi^{-1}\{y\}$ are pseudocompact, then the following are equivalent: ϕ is Z-closed; ϕ is zero set preserving; $g\in L(Y)$ for each $f\in L(X)$, where g is defined as in Lemma 2.

3. **Applications and examples.** Since for any space X and compact space Y, $\mathrm{pr}_1: X\times Y\to X$ is a closed mapping, an immediate corollary to Theorem 4 is that the product of two SW spaces, one of which is compact, is an SW space. Similarly one sees that the following holds.

THEOREM 6. *Let X and Y be SW spaces. Then $X\times Y$ is an SW space if and only if the identity mapping of $X\times Y$ onto $X\times_w Y$ is Z-closed.*

In [6] Tamano proves that if X and Y are pseudocompact completely regular spaces and X is a k-space, then $\mathrm{pr}_1: X\times Y\to X$ is Z-closed. The next theorem can be obtained from an obvious modification of his proof.

THEOREM 7. *If X and Y are SW spaces, one of which is a k-space, then $X\times Y$ is an SW space.*

In [2] a Hausdorff space X is said to be absolutely closed provided that for every Hausdorff space Y and continuous mapping $f: X\to Y$, $f(X)$ is a closed subset of Y. It is noted in [2] that a Hausdorff space X is absolutely closed if and only if every open filter base on X has an adherent point. We shall call a completely Hausdorff space X weakly absolutely closed if X has the following property: every open filter base on wX has an adherent point in X.

LEMMA 8. *Let X be a topological space, and let Y be a weakly absolutely closed space. If $f\in L(X\times Y)$, $a>0$, and $A=f^{-1}[0, a)$, then $\mathrm{pr}_1(\overline{A})$ is a closed subset of X.*

PROOF. Let $x_0\in[\mathrm{pr}_1(\overline{A})]^-$. Then for every neighborhood V of x_0, $(V\times Y)\cap\overline{A}$ and, consequently, $(V\times Y)\cap A$ are nonempty, so there is a point $x_V\in \mathrm{pr}_1((V\times Y)\cap A)$. Let $\mathcal{V}$ be the set of all neighborhoods of x_0, and for each $V\in\mathcal{V}$ denote $\bigcup\{f(x_W,\cdot)^{-1}[0,a) \mid W\subset V \text{ and } W\in\mathcal{V}\}$ by O_V. Since Y is weakly absolutely closed, $\{O_V \mid V\in\mathcal{V}\}$ has an adherent point y_0 in Y. Thus $(x_0, y_0)\in\overline{A}$ and $x_0\in\mathrm{pr}_1(\overline{A})$.

THEOREM 9. *If X is an SW space and Y is a weakly absolutely closed space, then $X\times Y$ is an SW space.*

PROOF. Since for a completely regular filter base $\mathfrak{F}$, $\bigcap\mathfrak{F}=\bigcap\{\overline{F}\mid F\in\mathfrak{F}\}$ = the adherence of $\mathfrak{F}$, a weakly absolutely closed space is an SW space. Thus X and Y are SW spaces.

Let $f\in L(X\times Y)$, $Z=f^{-1}\{0\}$, and for each $a>0$ denote $f^{-1}[0, a)$ by C_a. The set $D=\bigcap\{\mathrm{pr}_1(\overline{C}_a)\mid a>0\}$ is closed by Lemma 8. Furthermore, if $x_0 \in D$, then, by pseudocompactness, there must be a point y_0 at which $f(x_0, \cdot)$ vanishes. Thus $\mathrm{pr}_1(Z) = D$ is a closed subset of X.

Therefore, $X \times Y$ is an SW space by (i) of Theorem 4.

EXAMPLE 10. Clearly, an absolutely closed completely Hausdorff space is weakly absolutely closed. Thus if X is any SW space, and Y is an absolutely closed completely Hausdorff space, $X \times Y$ is an SW space. For examples of absolutely closed completely Hausdorff spaces which are not countably compact, see [1], [2], and [4]. In [5] a space is constructed which is a countably compact, noncompact, absolutely closed completely Hausdorff space. We now describe it in order to show that there exists a weakly absolutely closed space which is not a k-space.

Let Y be the "long interval," Ω the last point of Y, $\mathfrak{O}$ the order topology on Y, and $\mathfrak{I}$ the topology on Y which is generated by $\mathfrak{O} \cup \{ Y - L \mid L$ is the set of limit ordinals in $Y - \{\Omega\}\}$. The set $C = Y - \{\Omega\}$ is not a closed subset of $(Y, \mathfrak{I})$, but it can be shown that for every compact subset K of $(Y, \mathfrak{I})$, $C \cap K$ is compact.

It suffices to prove that if K is a compact subset of $(Y, \mathfrak{I})$ which contains Ω, then Ω is not a limit point of $(K, \mathfrak{I} \mid K)$. Since $Y - L \in \mathfrak{I}$ and K is compact, $(L \cap K, \mathfrak{I} \mid L \cap K)$ is compact. Since the identity mapping i: $(Y, \mathfrak{I}) \to (Y, \mathfrak{O})$ is continuous, it must also be true that $L \cap K$ is a compact subset of $(Y - \{\Omega\}, \mathfrak{O} \mid Y - \{\Omega\})$. Therefore, $\sup L \cap K < \Omega$. Appealing now to the countable compactness of $(K, \mathfrak{I} \mid K)$, one can conclude that $\sup K - \{\Omega\} < \Omega$, for if $\sup K - \{\Omega\}$ were equal to Ω, then one could find an increasing sequence $\{x_n\}$ in $K - \{\Omega\}$ such that $\sup L \cap K < \sup\{x_n\} \in L$.

EXAMPLE 11. In [5] (see also [4, Example 2]) it is noted that Tychonoff's regular but not completely regular space is an SW space. One can also prove that this space—call it X—is a k-space, because each of its points either has a compact neighborhood or a countable fundamental system of neighborhoods. According to Theorem 7, $X \times Y$ is an SW space for every SW space Y. X is not weakly absolutely closed, for if R is a Tychonoff plank in X and $\mathfrak{F}$ is the set of open rectangles in the upper right hand corner of R, $\mathfrak{F}$ has no adherent points.

EXAMPLE 12. Let $X = [0, 1]$, let $\mathfrak{I}$ = the usual topology on X, and choose disjoint dense subsets $X(1)$, $X(2)$, $X(3)$ of $(X, \mathfrak{I})$ such that $X = X(1) \cup X(2) \cup X(3)$. Let $\mathfrak{V}$ be the topology on X generated by $\mathfrak{I} \cup \{X(1), X(2)\}$.

In [4] Herrlich notes that $(X, \mathfrak{V})$ is a Urysohn-closed space which is not absolutely closed. One can also prove that $(X, \mathfrak{V})$ is weakly absolutely closed.

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