PRODUCT SPACES FOR WHICH THE STONE-WEIERSTRASS THEOREM HOLDS

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1. Introduction. A topological space $X$ is said to be a completely Hausdorff space (or a Stone space) provided that $C(X)$, the set of bounded continuous real valued functions defined on $X$, is point separating. In the following we call a completely Hausdorff space $X$ an SW space if every point separating subalgebra of $C(X)$ which contains the constants is uniformly dense in $C(X)$. Equivalently, according to Theorem 1, one can define a topological space to be an SW space provided that it can be obtained from a compact Hausdorff space by refining the topology without adding any continuous real valued functions.

In [5] the author proved that if $X$ and $Y$ are SW spaces, one of which is compact, then $X \times Y$ is an SW space. In this note a proof is given that if $X$ and $Y$ are SW spaces, then $X \times Y$ is an SW space if and only if $pr_1$ is $Z$-closed, i.e. for every zero set $Z$ of $X \times Y$, $pr_1(Z)$ is a closed subset of $X$. Several consequences of this theorem are considered, and some examples are given of noncompact spaces to which it applies.

We use the same terminology as that in [2] or [3]. Given a completely Hausdorff space $X$, we shall denote by $wX$ the completely regular space which has the same points and the same continuous real valued functions as those of $X$. $L(X)$ will denote the set of all functions in $C(X)$ which map $X$ into $[0, 1]$.

A filter base on a space $X$ is called an open filter base if the sets belonging to it are open subsets of $X$. An open filter base $\mathcal{F}$ is said to be completely regular if for each set $F \in \mathcal{F}$ there exist a set $G \in \mathcal{F}$ and a function $f \in L(X)$ such that $f$ vanishes on $G$ and equals 1 on $X - F$. $\mathcal{F}$ is said to be fixed if $\bigcap \mathcal{F} \neq \emptyset$.

Our proofs are all based on the following characterization theorem [1].

**Theorem 1 (Banaschewski).** Let $X$ be a completely Hausdorff space. The following are equivalent.

(i) $X$ is an SW space.

(ii) $wX$ is compact.

(iii) Every completely regular filter base on $X$ is fixed.

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2. A characterization theorem for products.

**Lemma 2.** Let \( \phi: X \to Y \) be a \( Z \)-closed open mapping from the space \( X \) onto the space \( Y \). Then, for any \( f \in L(X) \), the function \( g \), defined by \( g(y) = \inf \{ f(x) | \phi(x) = y \} \), belongs to \( L(Y) \).

**Proof.** One readily checks that for any \( s < t \),

\[
eg^{-1}(s, t) = \bigcup_{\lambda > s} Y - \phi(f^{-1}[0, \lambda]) \cap \phi(f^{-1}[0, t]),
\]

and it follows from the given properties of \( \phi \) that this set is open.

**Lemma 3.** If \( X \) is a completely Hausdorff space, \( Y \) an SW space, and \( \phi: X \to Y \) a \( Z \)-closed open surjection such that each \( \phi^{-1}\{y\}, y \in Y \), is an SW space, then \( X \) is also an SW space.

**Proof.** Let \( \mathcal{F} \) be a completely regular filter base on \( X \). Using Lemma 2, one can show that \( \phi(\mathcal{F}) \) is a completely regular filter base on \( Y \). Then there exists \( a \in \phi(\mathcal{F}) \) and for \( A = \phi^{-1}\{a\} \) a point \( b \in A \cap (\mathcal{F}) \), the latter since the restriction of \( \mathcal{F} \) to \( A \) is a completely regular filter base on \( A \).

**Theorem 4.** Let \( X \) and \( Y \) be SW spaces. The following are equivalent.

(i) \( \text{pr}_1 \) is \( Z \)-closed.

(ii) \( X \times Y \) is an SW space.

(iii) \( \text{w}(X \times Y) = \text{w}X \times \text{w}Y \).

**Proof.** (i) implies (ii). Since \( \{ f \circ \text{pr}_1 | i = 1 \text{ and } f \in \mathcal{C}(X), \text{ or } i = 2 \text{ and } f \in \mathcal{C}(Y) \} \) is a point separating subset of \( C(X \times Y) \), \( C(X \times Y) \) is point separating and \( X \times Y \) is a completely Hausdorff space. By Lemma 3, \( X \times Y \) is an SW space.

(ii) implies (iii). If \( X \times Y \) is an SW space, then \( \text{w}(X \times Y) \) is compact by Theorem 1. Since \( \text{w}X \times \text{w}Y \) is a Hausdorff space and the identity mapping \( i: \text{w}(X \times Y) \to \text{w}X \times \text{w}Y \) is a continuous bijection, \( i \) is a homeomorphism.

(iii) implies (i). (iii) implies that \( C(X \times Y) = C(\text{w}(X \times Y)) = C(\text{w}X \times \text{w}Y) \). Thus \( X \times Y \) and \( \text{w}X \times \text{w}Y \) have the same zero sets. Since \( \text{pr}_1: \text{w}X \times \text{w}Y \to \text{w}X \) is a closed mapping, and the closed subsets of \( \text{w}X \) are closed subsets of \( X \), (i) holds.

**Remark 5.** (i) of Theorem 4 holds if and only if \( \text{pr}_1 \) is zero set preserving. More generally, one can use Lemma 2 to prove that if \( \phi: X \to Y \) is an open surjection for which all \( \phi^{-1}\{y\} \) are pseudocom-
3. Applications and examples. Since for any space \( X \) and compact space \( Y \), \( \text{pr}_1: X \times Y \rightarrow X \) is a closed mapping, an immediate corollary to Theorem 4 is that the product of two SW spaces, one of which is compact, is an SW space. Similarly one sees that the following holds.

**Theorem 6.** Let \( X \) and \( Y \) be SW spaces. Then \( X \times Y \) is an SW space if and only if the identity mapping of \( X \times Y \) onto \( X \times wY \) is \( Z \)-closed.

In [6] Tamano proves that if \( X \) and \( Y \) are pseudocompact completely regular spaces and \( X \) is a \( k \)-space, then \( \text{pr}_1: X \times Y \rightarrow X \) is \( Z \)-closed. The next theorem can be obtained from an obvious modification of his proof.

**Theorem 7.** If \( X \) and \( Y \) are SW spaces, one of which is a \( k \)-space, then \( X \times Y \) is an SW space.

In [2] a Hausdorff space \( X \) is said to be absolutely closed provided that for every Hausdorff space \( Y \) and continuous mapping \( f: X \rightarrow Y \), \( f(X) \) is a closed subset of \( Y \). It is noted in [2] that a Hausdorff space \( X \) is absolutely closed if and only if every open filter base on \( X \) has an adherent point. We shall call a completely Hausdorff space \( X \) weakly absolutely closed if \( X \) has the following property: every open filter base on \( wX \) has an adherent point in \( X \).

**Lemma 8.** Let \( X \) be a topological space, and let \( Y \) be a weakly absolutely closed space. If \( f \in \mathcal{L}(X \times Y) \), \( a > 0 \), and \( A = f^{-1}[0, a) \), then \( \text{pr}_1(A) \) is a closed subset of \( X \).

**Proof.** Let \( x_0 \in \left[ \text{pr}_1(A) \right]^c \). Then for every neighborhood \( V \) of \( x_0 \), \( (V \times Y) \cap A \) and, consequently, \( (V \times Y) \cap \overline{A} \) are nonempty, so there is a point \( x_V \in \text{pr}_1((V \times Y) \cap A) \). Let \( \mathcal{U} \) be the set of all neighborhoods of \( x_0 \), and for each \( V \in \mathcal{U} \) denote \( U \left( f(x_V, \cdot) \right)^{-1} [0, a] \times W \) by \( O_V \). Since \( Y \) is weakly absolutely closed, \( \overline{\mathcal{O}_V} \) has an adherent point \( y_V \) in \( Y \). Thus \( (x_0, y_V) \in \overline{A} \) and \( x_0 \in \text{pr}_1(A) \).

**Theorem 9.** If \( X \) is an SW space and \( Y \) is a weakly absolutely closed space, then \( X \times Y \) is an SW space.

**Proof.** Since for a completely regular filter base \( \mathcal{F} \), \( \mathcal{F} = \cap \left\{ F \mid F \in \mathcal{F} \right\} \) is the adherence of \( \mathcal{F} \), a weakly absolutely closed space is an SW space. Thus \( X \) and \( Y \) are SW spaces. Let \( f \in \mathcal{L}(X \times Y) \), \( Z = f^{-1}[0, a] \), and for each \( a > 0 \) denote \( f^{-1}[0, a) \) by \( C_a \). The set \( D = \cap \left\{ \text{pr}_1(C_a) \mid a > 0 \right\} \) is closed by Lemma 8. Further-
more, if \( x_0 \in D \), then, by pseudocompactness, there must be a point \( y_0 \) at which \( f(x_0, \cdot) \) vanishes. Thus \( \text{pr}_1(Z) = D \) is a closed subset of \( X \).

Therefore, \( X \times Y \) is an SW space by (i) of Theorem 4.

Example 10. Clearly, an absolutely closed completely Hausdorff space is weakly absolutely closed. Thus if \( X \) is any SW space, and \( Y \) is an absolutely closed completely Hausdorff space, \( X \times Y \) is an SW space. For examples of absolutely closed completely Hausdorff spaces which are not countably compact, see [1], [2], and [4]. In [5] a space is constructed which is a countably compact, noncompact, absolutely closed completely Hausdorff space. We now describe it in order to show that there exists a weakly absolutely closed space which is not a \( k \)-space.

Let \( Y \) be the "long interval," \( \Omega \) the last point of \( Y \), \( \emptyset \) the order topology on \( Y \), and \( \mathfrak{F} \) the topology on \( Y \) which is generated by \( \emptyset \cup \{ Y - L | L \text{ is the set of limit ordinals in } Y - \{ \Omega \} \} \). The set \( C = Y - \{ \Omega \} \) is not a closed subset of \( (Y, \mathfrak{F}) \), but it can be shown that for every compact subset \( K \) of \( (Y, \mathfrak{F}) \), \( C \cap K \) is compact.

It suffices to prove that if \( K \) is a compact subset of \( (Y, \mathfrak{F}) \) which contains \( \Omega \), then \( \Omega \) is not a limit point of \( (K, \mathfrak{F}|K) \). Since \( Y - L \in \mathfrak{F} \) and \( K \) is compact, \( (L \cap K, \mathfrak{F}|L \cap K) \) is compact. Since the identity mapping \( i: (Y, \mathfrak{F}) \rightarrow (Y, \emptyset) \) is continuous, it must also be true that \( L \cap K \) is a compact subset of \( (Y - \{ \Omega \}, \emptyset|Y - \{ \Omega \}) \). Therefore, \( \sup L \cap K < \Omega \). Appealing now to the countable compactness of \( (K, \mathfrak{F}|K) \), one can conclude that \( \sup K - \{ \Omega \} < \Omega \), for if \( \sup K - \{ \Omega \} \) were equal to \( \Omega \), then one could find an increasing sequence \( \{x_n\} \) in \( K - \{ \Omega \} \) such that \( \sup L \cap K < \sup \{x_n\} \in L \).

Example 11. In [5] (see also [4, Example 2]) it is noted that Tychonoff's regular but not completely regular space is an SW space. One can also prove that this space—call it \( X \)—is a \( k \)-space, because each of its points either has a compact neighborhood or a countable fundamental system of neighborhoods. According to Theorem 7, \( X \times Y \) is an SW space for every SW space \( Y \). \( X \) is not weakly absolutely closed, for if \( R \) is a Tychonoff plank in \( X \) and \( \mathcal{F} \) is the set of open rectangles in the upper right hand corner of \( R \), \( \mathcal{F} \) has no adherent points.

Example 12. Let \( X = [0, 1] \), let \( \mathfrak{F} \) = the usual topology on \( X \), and choose disjoint dense subsets \( X(1), X(2), X(3) \) of \( (X, \mathfrak{F}) \) such that \( X = X(1) \cup X(2) \cup X(3) \). Let \( \mathcal{U} \) be the topology on \( X \) generated by \( \mathfrak{F} \cup \{ X(1), X(2) \} \).

In [4] Herrlich notes that \( (X, \mathcal{U}) \) is a Urysohn-closed space which is not absolutely closed. One can also prove that \( (X, \mathcal{U}) \) is weakly absolutely closed.
References


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