POSITIVE HARMONIC FUNCTIONS OF A BRANCHING PROCESS

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1. Introduction. Let \( \{p(n)\}_{n=0}^{\infty} \) be a sequence of positive numbers, one defines an infinite matrix \( p(x, y) \) \((x, y) \in \mathbb{N} \times \mathbb{N} \) as follows:

\[
p(0, y) = \delta(0, y), \quad p(1, y) = p(y);
\]

the other lines of the matrix are such that

\[
p(x + 1, y) = \sum_{y_1 + y_2 = y} p(1, y_1)p(x, y_2).
\]

We will say that the matrix \( p(x, y) \) is the branching matrix corresponding to the sequence \( \{p(n)\}_{n=0}^{\infty} \). When \( \sum_{n=0}^\infty p(n) = 1 \), the matrix is stochastic and describes a Markov process which is called a branching process.

We would like to get information about the positive harmonic functions \( h(x) \):

\[
\sum_y p(x, y)h(y) = h(x), \quad h(x) \geq 0.
\]

We will see that there is a one-to-one linear correspondence between these harmonic functions and those of a second triangular branching matrix. From that, it follows that there can be only one extreme harmonic function equal to 1 at \( x = 0 \). This function whenever it exists is an exponential function. We will be able to describe all the harmonic functions only when \( \sum_{n=0}^\infty p(n) = \sum_{n=0}^\infty np(n) = 1 \). This case has been already solved by Kesten, Ney and Spitzer [1].

2. Functional meaning of harmonic functions. Let \( h(x) \) be a positive function defined on \( \mathbb{N} \) and let \( p(x, y) \) be the branching matrix corresponding to a sequence \( \{p(n)\}_{n=0}^{\infty} \) of positive numbers, let us consider the cone of formal power series in a variable \( z \) with positive coefficients. On this cone, we define the following functional:

\[
H(g(z)) = \sum_{x=0}^{\infty} b(x)h(x) \quad \text{if} \quad g(z) = \sum_{x=0}^{\infty} b(x)z^x.
\]

If \( f(z) = \sum_{n=0}^\infty p(n)z^n \), we have that \( (f(z))^x = \sum_{y=0}^\infty p(x, y)z^y \). Since \( h(x) = H(z^x) \), \( h(x) \) is harmonic precisely when \( H(z^x) = H((f(z))^x) \), that is to say when \( H(g(z)) = H(g(f(z))) \) for every power series \( g(z) \).

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3. Transformation of harmonic functions.

**Theorem.** (a) If there is no number \( a \geq 0 \) such that \( \sum_{n=0}^{\infty} p(n)a^n \) converges to \( a \), there is no harmonic function different from 0.

(b) If \( a \) is the smallest positive solution of \( \sum_{n=0}^{\infty} p(n)a^n = 1 \), if \( p'(0) = 0 \) and

\[
p'(n) = \sum_{m=0}^{\infty} \binom{m}{n} p(m)a^{m-n} \quad (n \geq 1),
\]

and if \( p'(x, y) \) is the branching matrix corresponding to the sequence \( \{ p'(n) \} \), there is a linear one-to-one correspondence between positive harmonic functions of the matrix \( p(x, y) \) and those of the matrix \( p'(x, y) \).

Such a mapping is

\[
h'(x) = \sum_{n=0}^{\infty} \binom{x}{n} (-a)^{x-n} h(n), \quad h(x) = \sum_{n=0}^{\infty} \binom{x}{n} a^{x-n} h'(n).
\]

**Proof.** There is something to be proved only when \( p(0) \neq 0 \). Let \( f_n(z) \) be the functional product of the power series \( f(z) \) \( n \) times with itself. Let us start with a harmonic function \( h(x) \) for the matrix \( p(x, y) \). If \( g(z) = \sum_{n=0}^{\infty} b(x)z^n \) is a power series such that \( \sum_{n=0}^{\infty} |b(x)| h(x) < \infty \), we will set \( H(g(z)) = \sum_{n=0}^{\infty} b(x)h(x) \). Let \( n \) be an integer and \( t \) be a number between 0 and \( p(0) \), we set \( h_t(x) = H((z-t)x) \). Since \( f_n(z) - t \) is a power series with positive coefficients, we get that \( h_t(x) = 0 \).

(a) If \( \lim_{n \to \infty} f_n(0) = \infty \), then for every \( t \),

\[
h_t(x) = \sum_{n=0}^{\infty} \binom{x}{n} (-t)^{x-n} h(n) \geq 0.
\]

So for every \( x, h(x) = 0 \).

(b) If \( \lim_{n \to \infty} f_n(0) < \infty \), \( a = \lim_{n \to \infty} f_n(0) \). We will set \( h'(x) = H((z-a)^x) \).

Since \( h'(x) = \lim_{t \to 0} h_t(x) \), \( h'(x) \geq 0 \). On the cone of power series in a variable \( w \) \((w = z-a)\) with positive coefficients, \( h'(x) \) defines a positive functional \( H' \), where \( H'(w^x) = h'(x) \).

\[
H'((f(a+w) - a)^x) = H((f(z) - a)^x) = H((z-a)^x) = H'(w^x).
\]

Since \( f(a+w) - f(a) = \sum_{n=0}^{\infty} p'(n)w^n \), \( h'(x) \) is a harmonic function for the matrix \( p'(x, y) \).

Conversely, if \( h'(x) \) is a positive harmonic function for the matrix \( p'(x, y) \) and if \( H' \) is the corresponding functional on the power series in \( w \), we set \( h(x) = H'(a+w)^x \). From the change of variable \( w \to f(a+w) - a \), which leaves \( H' \) unchanged, we get that \( h(x) = H'((f(a+w))^x) \). So \( h(x) \) is also a harmonic function and
4. Applications. Let us remark that the matrix $p'(x, y)$ is triangular: if $y < x$, $p'(x, y) = 0$. On the diagonal $p'(x, x) = \mu x$ where $\mu = \sum_{m} m p(m) a^{m-1}$. $\mu$ is the derivative of $f(x)$ at $x=a$, so $\mu \leq 1$. If $p'(x)$ is an extreme function such that $h'(0) = 1$, since $\delta(0, x)$ is harmonic and from the fact that $h'(x) = [p'(x) - \delta(0, x)] + \delta(0, x)$ is the sum of two positive harmonic functions, then $h'(x) = \delta(0, x)$. The corresponding harmonic function $h(x)$ is $a^x$. This exponential is the only extreme harmonic function taking the value 1 at $x=0$ for the matrix $p(x, y)$. This last fact could have been proven directly. Let us sketch two other proofs for that.

Let $h(x)$ be an extreme harmonic function for $p(x, y)$ such that $h(0) = 1$.

(a) Let $h(x) = \min \{ h(y_1) h(y_2) \cdots h(y_x) | y_1 + y_2 + \cdots + y_x = x \}$; we will check that $h(x)$ is a superharmonic function. Let $w_1, w_2, \cdots, w_x$ be $x$ nonnegative integers whose sum is $x$, then

$$\sum_{y} p(x, y) h(y) = \sum_{w_1, w_2, \cdots, w_x} p(w_1, z_1) p(w_2, z_2) \cdots p(w_x, z_x) h(z_1 + z_2 + \cdots + z_x)$$

$$\leq \sum_{w_1, w_2, \cdots, w_x} p(w_1, z_1) \cdots p(w_x, z_x) h(z_1) \cdots h(z_x)$$

$$\leq h(w_1) h(w_2) \cdots h(w_x).$$

Hence $\sum_{y} p(x, y) h(y) \leq h(x)$.

Using the Riesz decomposition theorem for superharmonic function and noticing that in our case there is no potential $U(x)$ such that $U(0) = 1$, we get that $h(x)$ is harmonic and is equal to $h(x)$.

(b) $\delta(0, x)$ is a subharmonic function which is smaller than $h(x)$, so $h(x) = \lim_{n \to \infty} \sum_{0} p_n(x, y) \delta(0, y) = \lim_{n \to \infty} (f_n(0))^x$ where $p_n(x, y)$ is the branching process corresponding to the power series $f_n(x)$.

If $\mu = 1$ and if $h'(x)$ is a harmonic function such that $h'(1) = 0$, one can check easily that $h'(x) = 0$ if $x > 0$. So the only positive harmonic functions are in that case $A \delta(0, x) + B \delta(1, x)$ where $A \geq 0, B \geq 0$.

Reference


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