WHEN ARE MULTIPLICATIVE MAPPINGS ADDITIVE?

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Let R and S be arbitrary associative rings (not necessarily with identity elements). A one-one mapping σ of R onto S such that $(xy)^\sigma = x^\sigma y^\sigma$ for all $x, y \in R$ will be called a multiplicative isomorphism of R onto S. The question of when a multiplicative isomorphism is additive has been considered by Rickart [1] and also by Johnson [2]. In both of these papers some sort of minimality conditions were imposed on the ring R. It is our aim in this note to generalize the main theorem of Rickart's paper [1, p. 761, Theorem II] and at the same time remove the minimality condition. (Our results are along different lines than those in Johnson's paper [2], in which he assumes each nonzero "closed" right ideal contains a minimal nonzero "closed" right ideal.) Rickart's theorem says the following:

Let R be a ring containing a family $\{J_\alpha \mid \alpha \in A\}$ of right ideals which satisfies

(i) Each J_α is irreducible (i.e., J_α is minimal and $J_\alpha R \neq 0$).

(ii) $J_\alpha x = 0$ for each $\alpha \in A$, implies $x = 0$ (hence R is Jacobson semisimple).

(iii) Each J_α, considered as a vector space over the division ring $\mathrm{Hom}_R(J_\alpha, J_\alpha)$, is of dimension greater than one.

Then any multiplicative isomorphism of R onto an arbitrary ring S is necessarily additive.

It is well known that any minimal right ideal in a semisimple ring is of the form $J = eR$, e an idempotent. Semisimplicity also says that $xR = 0$ implies $x = 0$. From (iii) we may conclude that for each J $(= J_\alpha)$ there is a nonzero "vector" in $J = eR$ of the form $ey(1-e)$. Indeed, if $eR(1-e) = 0$ then $eR = eRe$, which says that J is one dimensional over the division ring eRe. Therefore, if a "scalar" exe is such that $(exe)(eR)(1-e) = 0$ then $exe = 0$.

The remarks in the preceding paragraph show that Rickart's result is a special case of our main theorem, which we now state.

THEOREM. *Let R be a ring containing a family $\{e_\alpha \mid \alpha \in A\}$ of idempotents which satisfies*:

(1) $xR = 0$ *implies* $x = 0$.

(2) *If* $e_\alpha R x = 0$ *for each* $\alpha \in A$, *then* $x = 0$ (*and hence* $Rx = 0$ *implies* $x = 0$).

(3) *For each* $\alpha \in A$, $e_\alpha x e_\alpha R(1-e_\alpha) = 0$ *implies* $e_\alpha x e_\alpha = 0$.

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Then any multiplicative isomorphism σ of R onto an arbitrary ring S is additive.

The proof will be organized in a series of lemmas. We begin with the trivial

LEMMA 1. $0^\sigma = 0$.

PROOF. Since σ is onto, $x^\sigma = 0$ for some $x \in R$. Then $0^\sigma = (0 \cdot x)^\sigma = 0^\sigma \cdot x^\sigma = 0$.

For the next several lemmas we will be just dealing with one fixed idempotent e_α of the family. We call this idempotent e_1 and formally set $e_2 = 1 - e_1$ (R need not have an identity element). Then, letting $R_{ij} = e_i R e_j$, $i, j = 1, 2$, we may write R in its Peirce decomposition $R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$. x_{ij} will denote an element of R_{ij}.

LEMMA 2. $(x_{ii} + x_{jk})^\sigma = x_{ii}^\sigma + x_{jk}^\sigma$, $j \neq k$.

PROOF. First assume that $i = j = 1$ and $k = 2$. We may find an element z of R such that $z^\sigma = x_{11}^\sigma + x_{12}^\sigma$, since σ is onto. For $a_{1j} \in R_{1j}$ we have

$$\begin{aligned}(za_{1j})^\sigma &= z^\sigma a_{1j}^\sigma = (x_{11}^\sigma + x_{12}^\sigma)a_{1j}^\sigma = x_{11}^\sigma a_{1j}^\sigma + x_{12}^\sigma a_{1j}^\sigma = (x_{11}a_{1j})^\sigma + (x_{12}a_{1j})^\sigma \\ &= [(x_{11} + x_{12})a_{1j}]^\sigma + 0^\sigma = [(x_{11} + x_{12})a_{1j}]^\sigma.\end{aligned}$$

Therefore $za_{1j} = (x_{11} + x_{12})a_{1j}$, since σ is one-one. In the same fashion, for $a_{2j} \in R_{2j}$, we have $(za_{2j})^\sigma = z^\sigma a_{2j} = (x_{11}a_{2j}^\sigma)^\sigma + (x_{12}a_{2j})^\sigma = [(x_{11} + x_{12})a_{2j}]^\sigma$, yielding $za_{2j} = (x_{11} + x_{12})a_{2j}$. We have thus shown that $[z - (x_{11} + x_{12})]R = 0$, and so, by condition (1), we see that $z = x_{11} + x_{12}$, i.e. $x_{11}^\sigma + x_{12}^\sigma = (x_{11} + x_{12})^\sigma$. The only essentially different choice for i, j, k is to let $i = k = 1$ and let $j = 2$. In this case we are led to $R[z - (x_{11} + x_{21})] = 0$, and so once again $z = x_{11} + x_{21}$ in view of condition (2).

LEMMA 3. *σ is additive on R_{12}.*

PROOF. Let x_{12}, $u_{12} \in R_{12}$ and choose $z \in R$ such that $z^\sigma = x_{12}^\sigma + u_{12}^\sigma$. For $a_{1j} \in R_{1j}$ we have $(za_{1j})^\sigma = z^\sigma a_{1j}^\sigma = (x_{12}^\sigma + u_{12}^\sigma)a_{1j}^\sigma = (x_{12}a_{1j})^\sigma + (u_{12}a_{1j})^\sigma = 0$, whence $za_{1j} = 0$. For $a_{2j} \in R_{2j}$, we see that

$$\begin{aligned}(za_{2j})^\sigma &= z^\sigma a_{2j}^\sigma = (x_{12}^\sigma + u_{12}^\sigma)a_{2j}^\sigma = (e_1^\sigma + x_{12}^\sigma)(a_{2j}^\sigma + u_{12}^\sigma a_{2j}^\sigma) \\ &= (e_1^\sigma + x_{12}^\sigma)[a_{2j}^\sigma + (u_{12}a_{2j})^\sigma] = (e_1 + x_{12})^\sigma(a_{2j} + u_{12}a_{2j})^\sigma \\ &= [(e_1 + x_{12})(a_{2j} + u_{12}a_{2j})]^\sigma = [(x_{12} + u_{12})a_{2j}]^\sigma,\end{aligned}$$

making use of Lemma 2. Hence $za_{2j} = (x_{12} + u_{12})a_{2j}$. It follows that $[z - (x_{12} + u_{12})]R = 0$, and so by condition (1), $z = x_{12} + u_{12}$.

LEMMA 4. *σ is additive on R_{11}.*

PROOF. Let x_{11}, $u_{11}\in R_{11}$ and write $z^\sigma = x_{11}^\sigma + u_{11}^\sigma$ for some $z\in R$. Using Lemma 3 we see that $z^\sigma a_{12}^\sigma = x_{11}^\sigma a_{12}^\sigma + u_{11}^\sigma a_{12}^\sigma = (x_{11}a_{12})^\sigma + (u_{11}a_{12})^\sigma = (x_{11}a_{12}+u_{11}a_{12})^\sigma$, where $a_{12}\in R_{12}$. This shows that $za_{12} = (x_{11}+u_{11})a_{12}$, in other words, $[z-(x_{11}+u_{11})]R_{12}=0$. Next we write z in terms of its components $z = z_{11}+z_{12}+z_{21}+z_{22}$ and note that

$$\begin{aligned} z^\sigma &= x_{11}^\sigma + u_{11}^\sigma = (e_1x_{11})^\sigma + (e_1u_{11})^\sigma = e_1^\sigma x_{11}^\sigma + e_1^\sigma u_{11}^\sigma = e_1^\sigma(x_{11}^\sigma + u_{11}^\sigma) \\ &= e_1^\sigma(z_{11} + z_{12} + z_{21} + z_{22})^\sigma = [e_1(z_{11} + z_{12} + z_{21} + z_{22})]^\sigma = (z_{11} + z_{12})^\sigma. \end{aligned}$$

These equations show that $z = z_{11}+z_{12}$, whence $z_{21}=z_{22}=0$. By repeating the argument with e_1 multiplied in on the right, one finds that $z_{12}=0$, thus yielding $z=z_{11}\in R_{11}$. Therefore $z-(x_{11}+u_{11})\in R_{11}$ and our previous conclusion that $[z-(x_{11}+u_{11})]R_{12}=0$ forces $z = x_{11}+u_{11}$ because of condition (3).

LEMMA 5. *σ is additive on $e_1R = R_{11}+R_{12}$.*

PROOF. Let x_{11}, $u_{11}\in R_{11}$ and let x_{12}, $u_{12}\in R_{12}$. Then Lemmas 2, 3, and 4 are all used in seeing that the equations

$$\begin{aligned} [(x_{11} + x_{12}) + (u_{11} + u_{12})]^\sigma &= [(x_{11} + u_{11}) + (x_{12} + u_{12})]^\sigma \\ &= (x_{11} + u_{11})^\sigma + (x_{12} + u_{12})^\sigma \\ &= x_{11}^\sigma + u_{11}^\sigma + x_{12}^\sigma + u_{12}^\sigma \\ &= (x_{11} + x_{12})^\sigma + (u_{11} + u_{12})^\sigma \end{aligned}$$

hold true.

PROOF OF THE THEOREM. Let x, $y\in R$ and write $z^\sigma = x^\sigma + y^\sigma$ for some $z\in R$. For $\alpha\in A$, select any $t_\alpha\in e_\alpha R$. Then $(t_\alpha z)^\sigma = t_\alpha^\sigma z^\sigma = t_\alpha^\sigma(x^\sigma+y^\sigma) = t_\alpha^\sigma x^\sigma + t_\alpha^\sigma y^\sigma = (t_\alpha x)^\sigma + (t_\alpha y)^\sigma = (t_\alpha x + t_\alpha y)^\sigma$, since σ is additive on $e_\alpha R$ by Lemma 5. Hence $t_\alpha z = t_\alpha(x+y)$, and so we have proved that $e_\alpha R[z-(x+y)]=0$ for all $\alpha\in A$. Condition (2) may then be invoked to conclude that $z=x+y$. This says that $(x+y)^\sigma = x^\sigma + y^\sigma$, and the theorem is proved.

COROLLARY. *Let R be a prime ring containing an idempotent $e\neq 0, 1$ (R need not have an identity). Then any multiplicative isomorphism of R onto an arbitrary ring S is additive.*

COROLLARY. *Let R satisfy the conditions of the theorem (or the preceding corollary). Then any multiplicative anti-isomorphism ϕ of R onto an arbitrary ring S is additive.*

PROOF. Let τ be the anti-isomorphism of S onto the opposite ring S^* of S. By the theorem $\sigma = \tau\phi$ is an additive mapping of R onto S^*, and so ϕ is additive.

An interesting feature of this problem is that the conclusion of the theorem obviously fails if the ring R is either too "well behaved" or too "badly behaved." Indeed, if R is a field, the mapping $x \to x^{-1}$ (with $0 \to 0$) is not in general additive. Hence the need for condition (3). On the other hand, if $R^2 = 0$, any one-one mapping of the set R onto itself (with $0 \to 0$) is multiplicative. Conditions (1) and (2) prevent occurrences of this sort.

We remark finally that the condition that σ be onto appears to be important. Indeed, let $R = F_2$ and let $S = F_3$, where F_n denotes the ring of $n \times n$ matrices over the field F. If $a \in R$, then the mapping

$$a \to \begin{pmatrix} a & 0 \\ 0 & \det a \end{pmatrix}$$

is a one-one multiplicative mapping of R into S which is clearly not additive.

REFERENCES

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