If $S$ is a coherent analytic sheaf on the complex analytic space $X$, then for each $x \in X$, the stalk $S(x)$ is a finitely generated $\mathcal{O}(x)$-module, where $\mathcal{O}$ is the structure sheaf of $X$ [1]. Since $\mathcal{O}(x)$ is a local ring, there is a minimum number, $\#(S, x)$, of germs that generate $S(x)$ as an $\mathcal{O}(x)$-module, and every set of generators for $S(x)$ contains a subset of $\#(S, x)$ generators [2, p. 14].

If there are $n$ global sections $s_1, \ldots, s_n \in S(X)$ whose germs generate the stalk of $S$ at every point, then evidently:

(A) for every $x \in X$, $S(x)$ is generated by global sections of $S$ and,

(B) $\{\#(S, x) : x \in X\}$ is a bounded set of integers. In fact, $\{\#(S, x) : x \in X\}$ is bounded by $n$. The principal result of this note is that the converse also is true in case $X$ has finite global dimension.

If the global sections of $S$ generate its stalk at each point and if $\{\#(S, x) : x \in X\}$ is bounded, then finitely many of the global sections of $S$ generate its stalk at each point.

Let us say that a subset $G$ of $S(X)$ generates $S|K$ if for each $x \in K$, $\{s(x) : s \in G\}$ generates the stalk $S(x)$. If $K = X$, say that $G$ generates $S$. An ordered $n$-tuple $(s_1, \ldots, s_n) \in S(X^n)$ generates $S|K$ if $\{s_1, \ldots, s_n\}$ generates $S|K$. Let $G(S, n, K)$ be the set of all $n$-tuples in $S(X)^n$ which generate $S|K$.

If $U$ is an open subset of $X$, then $S(U)$ has a natural metrizable topology, which makes $S(U)$ into a Fréchet space. If $V$ is open and contains $U$, the restriction map $r_{UV}: S(V) \rightarrow S(U)$ is continuous [1, Chapter VIII]. A residual set in $S(X)^n$ is the complement of a set of the first category.

1. **Theorem.** Let $X$ be a $d$-dimensional analytic space and let $S$ be a coherent analytic sheaf on $X$ that is generated by $S(X)$. If $\#(S, x) \leq n$ for every $x \in X$, then $G(S, n(d+1), X)$ is a dense residual set in $S(X)^{n(d+1)}$; in particular, it is not empty.

The theorem follows from a series of lemmas.

2. **Lemma.** Let $X$ be a complex analytic space and let $S$ be a coherent analytic sheaf on $X$. If $U$ is an Oka-Weil domain in $X$ [1, p. 211] and $K$ is a compact $\mathcal{O}(U)$-convex subset of $U$, then $G(S, n, K)$ is open in $S(X)^n$, for each positive integer $n$.

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Proof. Suppose \( t_1, \ldots, t_n \in \mathcal{S}(X) \) generate \( \mathcal{S} \mid K \). Then \( K \) has an open neighborhood \( V \subseteq U \) that is also an Oka-Weil domain such that \( t_1, \ldots, t_n \) generate \( \mathcal{S} \mid V \) [1, pp. 211 and 244]. Identify \( \mathcal{O}^{n \times n} \) with the space of \( n \times n \) complex matrices, and let \( E: \mathcal{O}^{n \times n} \mid V \to \mathcal{S} \mid V \) be the map defined by \( E(f) = (\sum_j f_{ij} t_j, \ldots, \sum_j f_{nj} t_j) \). Then \( E \) is a homomorphism of coherent sheaves. \( \text{Ker } E \) is a coherent sheaf, and by Cartan’s Theorem B, \( H^1(V, \text{Ker } E) = 0 \). Since \( t_1, \ldots, t_n \) generate \( \mathcal{S} \mid V \), the sequence \( 0 \to H^0(V, \text{Ker } E) \to H^0(V, \mathcal{O}^{n \times n}) \to H^0(V, \mathcal{S}) \to 0 \) is exact. That is, \( E: \mathcal{O}(V)^{n \times n} \to \mathcal{S}(V) \) is a surjection. \( R = \{ f \in \mathcal{O}(V)^{n \times n} : f(x) \text{ is an invertible matrix for each } x \in K \} \) is open in \( \mathcal{O}(V)^{n \times n} \), since the topology of \( \mathcal{O}(V)^{n \times n} \) is that of uniform convergence on compacta [1, p. 237], and the set of invertible matrices is open in \( \mathcal{O}^{n \times n} \). It follows from the Open Mapping Theorem for Fréchet spaces that \( R(E) \) is open in \( \mathcal{S}(V)^n \). Since \( r \mathcal{S} \mathcal{V} \) is continuous, \( r \mathcal{S} \mathcal{V}^{-1}(E(R)) \) is open in \( \mathcal{S}(X)^n \). But if \( s \in \mathcal{S}(X)^n \) and \( r \mathcal{S} \mathcal{V}(s) \in E(R) \), then \( s \) generates \( \mathcal{S} \mid V \). Thus \( G(\mathcal{S}, n, K) \) contains a neighborhood \( r \mathcal{S} \mathcal{V}(E(R)) \) of \( t \).

3. Lemma. Let \( X \) be a complex analytic space, \( \mathcal{S} \) a coherent analytic sheaf on \( X \), and \( s_1, \ldots, s_n \in \mathcal{S}(X) \). Then \( Y = \{ y \in X : s_1(y), \ldots, s_n(y) \} \) do not generate \( \mathcal{S}(y) \} \) is an analytic subvariety of \( X \).

Proof. Let \( 3 \) be the subsheaf of \( \mathcal{S} \) generated by \( s_1, \ldots, s_n \). Then \( Y \) is the support of the coherent analytic sheaf \( \mathcal{S}(X)^n/3 \) [3, p. 87].

4. Lemma. Let \( X \) be a complex analytic space, let \( x \in X \), and let \( \mathcal{S} \) be a coherent analytic sheaf on \( X \) such that \( \mathcal{S}(X) \) generates \( \mathcal{S}(x) \). Let \( n \geq \#(\mathcal{S}, x) \). Then \( G(\mathcal{S}, n, \{ x \}) \) is dense in \( \mathcal{S}(X)^n \).

Proof. Since \( \mathcal{S}(X) \) generates \( \mathcal{S}(x) \) and \( n \geq \#(\mathcal{S}, x) \), we can choose a \( t \in \mathcal{S}(X)^n \) that generates \( \mathcal{S}(x) \). Let \( s \) be any element of \( \mathcal{S}(X)^n \). Say \( s_t(x) = \sum_j c_{ij} t_j(x) \), where the matrix-valued function \( c \) is analytic in a neighborhood of \( x \). Then \( s(x) - \lambda t(x) = (c - \lambda t(x)) \), so that \( s - \lambda t \) will generate \( \mathcal{S}(x) \) provided that the matrix \( c - \lambda t \) is nonsingular in a neighborhood of \( x \). This will be true if \( \lambda \) is distinct from each of the \( n \) eigenvalues of the matrix \( c(x) \). There are arbitrarily small numbers \( \lambda \) with this property. Hence there are sections \( s - \lambda t \) of \( \mathcal{S}(X)^n \) arbitrarily close to \( s \) that generate \( \mathcal{S}(x) \).

5. Lemma. Let \( X \) be a \( d \)-dimensional complex analytic space, let \( \mathcal{S} \) be a coherent analytic sheaf on \( X \), and let \( K \) be a compact subset of \( X \). Suppose that \( \mathcal{S}(X) \) generates \( \mathcal{S} \mid K \) and that \( n \geq \#(\mathcal{S}, x) \) for each \( x \in K \). Then \( G(\mathcal{S}, n(d + 1), K) \) is dense in \( \mathcal{S}(X)^{n(d + 1)} \).

Proof. Let \( A \) be a nonempty open set in \( \mathcal{S}(X)^{n(d + 1)} \). Then \( A \) contains a nonempty open set of the form \( A_1 \times \cdots \times A_{d+1} \), where \( A_i \)
is open in \( s(X)^n \) for \( i = 1, \ldots, d+1 \). Suppose that \( 0 \leq k \leq d+1 \). Let us show that for each \( i \) such that \( 1 \leq i \leq k \), we can choose a section \( s^i \in A_i \) with the following property. Let \( Y_k = \{ x \in X : (s^1(x), \ldots, s^k(x)) \) does not generate \( s(x) \} \). (\( Y_k \) is a variety by Lemma 3.) Then no irreducible branch of \( Y_k \) of dimension greater than \( d - k \) intersects \( K \).

The proof is by induction on \( k \). For \( k = 0 \), \( Y_0 = X \) and there is nothing to prove. Suppose we have chosen \( s^1, \ldots, s^k \) so that no irreducible branch of \( Y_k \) of dimension greater than \( d - k \) intersects \( K \). Let \( B_1, \ldots, B_p \) be the irreducible branches of \( Y_k \) which do intersect \( K \). Since \( S \) is coherent, there is actually a neighborhood \( U \) of \( K \) such that \( \#(S, x) \leq n \) for each \( x \in U \). Therefore, for each \( j = 1, \ldots, p \), we can choose a regular point \( x_j \in B_j \) such that \( \#(S, x_j) \leq n \). Then \( D_j = \{ s \in S \in X^n : s \text{ generates } S(x_j) \} \) is open (Lemma 2) and dense (Lemma 4) in \( S(X)^n \). Choose \( s^{k+1} \) in \( A_{k+1} \cap D_1 \cap \cdots \cap D_p \). Then no \( (n - k) \)-dimensional branch of \( Y_{k+1} \) can intersect \( K \).

In particular, \( (s^1, \ldots, s^{d+1}) \) is an element of \( A \) such that \( Y_{d+1} \cap K = \emptyset \), or in other words, \( (s^1, \ldots, s^{d+1}) \) is an element of \( A \) that generates \( S | K \).

**Proof of Theorem 1.** Express \( X \) as the union of countably many compact subsets \( K_1, K_2, K_3, \ldots \), each of which is contained in an Oka-Weil domain in which it is holomorphically convex. According to Lemmas 2 and 5, \( G(S, n(d+1), K_j) \) is open and dense in \( S(X)^n(d+1) \) for each \( j = 1, 2, 3, \ldots \). Therefore,

\[
G(S, n(d+1), X) = \bigcap_{j=1}^\infty G(S, n(d+1), K_j)
\]

is a residual set in \( S(X)^n(d+1) \). Since \( S(X)^n(d+1) \) is a Fréchet space, the Baire Category Theorem shows that \( G(S, n(d+1), X) \) is dense in \( S(X)^n(d+1) \).

**6. Corollary.** If \( X \) is a \( d \)-dimensional analytic space and \( B \) is a \( n \)-dimensional vector bundle over \( X \) which is generated by its global sections, then \( B \) is generated by \( n(d+1) \) of its global sections.

If \( X \) is a Stein space, every coherent sheaf on \( X \) satisfies condition \( (A) \), according to Cartan's Theorem A [1]. At least when \( X \) is an open subset of a Stein manifold, the converse is also true. Indeed, in this case \( X \) satisfies the hypotheses of the following proposition, according to Rossi [4].

**7. Proposition.** Let \( X \) be an analytic space with the following properties.
(a) $X$ can be embedded as an open subset of a Stein space $Y$ in such a way that the restriction map $r: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is bijective.

(b) Whenever $\mathcal{S}$ is the sheaf of ideals of a $0$-dimensional variety in $X$, $\mathcal{S}(X)$ generates $\mathcal{S}$.

Then $X$ is a Stein space.

Proof. It will be enough to show that $X = Y$. If $X \neq Y$, there must be a component $C$ of $Y$ which is not contained in $X$. However, $C \cap X$ cannot be empty since the restriction $\mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is injective. Therefore, there must be a $y$ on the boundary of $C \cap X$ with respect to $C$.

Let $\{x_n\}$ be a sequence in $C \cap X$ converging to $y$, and let $\mathcal{S}$ be the sheaf of ideals of $\{x_n\}$ over $X$. Let $Z = \{x \in X : f(x) = 0 \text{ for each } f \in \mathcal{S}(X)\}$. It will be enough to show that $Z$ has dimension greater than 0, for then we will have contradicted the hypothesis that $\mathcal{S}$ is generated by its global sections. Since $r$ is a bijection, we can form $W = \{w \in Y : (r^{-1}f)(w) = 0 \text{ for each } f \in \mathcal{S}(X)\}$. Clearly, $W$ is a subvariety of $Y$ and $W \cap X = Z$. Since $y$ is an accumulation point of $Z$, dim$_y(W) \neq 0$. If the irreducible branches of $W$ passing through $y$ are $B_1, \cdots, B_p$, at least one of them must contain infinitely many points of $Z$, $B_1$ let us say. But then dim $B_1 > 0$, and dim $Z \geq$ dim $B_1$.

8. Proposition. Let $M$ be a $2$-dimensional complex manifold and let $f: \mathcal{O}^n \rightarrow \mathcal{O}^m$ be an $\mathcal{O}$-homomorphism. Then for every $x \in M$, #(Ker $f$, $x$) $\leq n$.

Proof. Consider the exact sequence $(\text{Ker } f)(x) \rightarrow \mathcal{O}(x)^n \rightarrow \mathcal{O}(x)^m \rightarrow (\text{Im } f)(x) \rightarrow 0$. According to the Hilbert Syzygy Theorem [1, p. 74], $(\text{Ker } f)(x)$ is a free $\mathcal{O}(x)$-module. Since $(\text{Ker } f)(x)$ is a free submodule of $\mathcal{O}(x)^n$, #(Ker $f$, $x$) $\leq n$. (The field of quotients $Q$ of $\mathcal{O}(x)$, being the direct limit of copies of $\mathcal{O}(x)$, is a flat $\mathcal{O}(x)$-module. Thus the sequence $0 \rightarrow Q \otimes_{\mathcal{O}(x)} (\text{Ker } f)(x) \rightarrow Q \otimes_{\mathcal{O}(x)} \mathcal{O}(x)^n$ is exact. But $Q \otimes_{\mathcal{O}(x)} \mathcal{O}(x)^n$ is an $n$-dimensional vector space over $Q$, and $Q \otimes (\text{Ker } f)(x)$ is a vector space over $Q$ of dimension #(Ker $f$, $x$).)

9. Corollary. Let $M$ be a $2$-dimensional Stein manifold. If $I$ and $J$ are finitely generated ideals in the ring of holomorphic complex-valued functions on $M$, then $I \cap J$ is also finitely generated.

Proof. The following proof was suggested to me by Lance Small. Let $a_1, \cdots, a_n$ be generators for $I$, and let $b_1, \cdots, b_m$ be generators for $J$. Let $\mathcal{S}$ and $\mathcal{G}$ be respectively the subsheaves of $\mathcal{O}$ generated by $I$ and $J$. According to Cartan's Theorem B, $I = \mathcal{S}(M)$, $J = \mathcal{G}(M)$, and $I \cap J = (\mathcal{S} \cap \mathcal{G})(M)$. Thus it will suffice to prove that $\mathcal{S} \cap \mathcal{G}$ is generated by finitely many of its global sections. In fact, it will be shown that
#(g \cap J, x) \leq n + m$ for every $x \in M$, and hence that $I \cap J$ is generated by $3(n + m)$ of its elements according to Theorem 1.

Let $f: \bigoplus_{n+m} \to \emptyset$ be the map defined by $f(c_1, \cdots, c_{n+m}) = c_1 a_1 + \cdots + c_n a_n - c_{n+1} b_1 - \cdots - c_{n+m} b_m$. Then $\#(\text{Ker } f, x) \leq n + m$ for every $x \in M$ by Proposition 8. But the formula $\pi(c_1, \cdots, c_{n+m}) = c_1 a_1 + \cdots + c_n a_n$ evidently defines a surjection $\pi: \text{Ker } f \to g \cap J$. It follows that $\#(g \cap J, x) \leq \#(\text{Ker } f, x) \leq n + m$ for every $x \in M$.

I do not know whether Corollary 9 would remain true if the condition of 2-dimensionality were dropped. I conjecture that it would not. In one dimension, the corollary is trivial since every finitely generated ideal is then principal.

10. **Example.** The bound $n(d + 1)$ is the best possible if the density of $G(S, n(d + 1), X)$ in $\mathcal{O}(X)^{n(d+1)}$ is to be preserved.

Consider, in the complex plane $\mathcal{C}$, the subsheaf $S$ of $\mathcal{O}$ generated by the coordinate function $z$. Although one global section suffices to generate $S$, every section in a neighborhood of the section $z^2$ has two zeros in $\mathcal{C}$, and therefore fails to generate $S$.

**References**


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