ON THE GAPS IN THE SPECTRUM ASSOCIATED WITH HILL'S EQUATION

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1. Let $q(x)$ be a real-valued, piecewise continuous, periodic function with period a and let S denote the spectrum associated with the equation

$$\psi''(x) + \{\lambda - q(x)\}\psi(x) = 0, \tag{1.1}$$

holding for $-\infty < x < \infty$. Any gap in S with midpoint μ has length not exceeding

$$2\left(\frac{1}{a}\int_0^a q^2(x)dx\right)^{1/2} \tag{1.2}$$

if $\mu \geqq 0$. This was first proved by Putnam [3] and later obtained as a special case of a more general result [2, Theorem 3]. Putnam queried the necessity of the condition $\mu \geqq 0$ and raised the question of whether all gaps in S do not exceed (1.2) in length. In this connexion, he pointed out that, if $q(x)$ has mean value zero, i.e. if

$$\int_0^a q(x)dx = 0, \tag{1.3}$$

then all gaps in S do not exceed (1.2) in length if

$$\int_0^a q^2(x)dx \leqq 256a^{-3}, \tag{1.4}$$

inasmuch as that this condition automatically implies that $\mu \geqq 0$ for all gaps. Actually, Putnam worked with $a = 1$, but the results for general a are obtained by the transformation $x = at$.

In this paper I answer Putnam's question in the negative in §2 by constructing a $q(x)$ satisfying (1.3) such that the first gap in S has length exceeding (1.2). I mention here for comparison with (1.4) that my $q(x)$ satisfies

$$\int_0^a q^2(x)dx > 2^{10}\pi^4 256a^{-3}. \tag{1.5}$$

Then, in §3, I give an extension of the result concerning (1.2).

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2. Let μ_n and Λ_n be the eigenvalues of (1.1) for the interval $0 \leqq x \leqq a$ with the boundary conditions

$$\psi(0) = -\psi(a), \qquad \psi'(0) = -\psi'(a),$$

and

$$\psi(0) = \psi(a) = 0$$

respectively. Then the first gap in S is the interval (μ_0, μ_1), and our aim is to construct a $q(x)$ for which

$$\mu_0 + 2\left(\frac{1}{a}\int_0^a q^2(x)dx\right)^{1/2} < \mu_1. \tag{2.1}$$

Let α, b, c be positive numbers with $2b+2c \leqq a$ and let

$$\begin{aligned} q(x) &= -\alpha && 0 \leqq x \leqq b, \\ &= 0 && b < x \leqq b+c, \\ &= \alpha && b+c < x \leqq 2b+c, \\ &= 0 && 2b+c < x \leqq a. \end{aligned} \tag{2.2}$$

Clearly (1.3) is satisfied. We now obtain estimates for μ_0 and μ_1 and impose conditions on α, b, c as we go along. The simplest way to deal with μ_1 is to use the inequality $\mu_1 \geqq \Lambda_0$ [**1**, p. 215 (3.15)]. Let

$$\alpha b^2 \leqq \tfrac{1}{4}\pi^2. \tag{2.3}$$

Then, if Λ_0' corresponds to Λ_0 in the problem with $q(x)$ replaced by $q_1(x)$, where

$$\begin{aligned} q_1(x) &= -\alpha && (0 \leqq x \leqq b), \\ &= 0 && (b < x \leqq a), \end{aligned}$$

we have $\Lambda_0' > 0$ [**4**, §5.5]. Since $q(x) \geqq q_1(x)$, $\Lambda_0 \geqq \Lambda_0'$ and therefore

$$\mu_1 > 0. \tag{2.4}$$

Next,

$$\mu_0 = \inf\left(\int_0^a [\{f'(x)\}^2 + q(x)f^2(x)]dx \Big/ \int_0^a f^2(x)dx\right),$$

the inf being taken over all continuous $f(x)$ with piecewise continuous derivative such that $f(0) = -f(a)$. Choose $f(x)$ to be the function whose graph consists of straight line segments joining the points $(0, 1)$, $(b, 1)$, $(b+c, 0)$, $(a-c, 0)$, $(a, -1)$. Then, as is easily verified,

$$\mu_0 \leqq \frac{2/c - \alpha b}{2c/3 + b} = \frac{1}{b^2}\,\frac{2d - \alpha b^2}{2/3d + 1},$$

where $d = b/c$. Hence, by (2.4), (2.1) is satisfied if

$$\frac{1}{b^2}\,\frac{2d - \alpha b^2}{2/3d + 1} + 2(2\alpha^2 b/a)^{1/2} \leqq 0,$$

i.e. if $a^{1/2} \geqq 2\sqrt{2}\alpha b^{5/2}(2/3d+1)/(\alpha b^2 - 2d)$. The best choice of d would be the one between 0 and $\frac{1}{2}\alpha b^2$ which minimizes the right-hand side. A convenient choice, however, is $d = \frac{1}{4}\alpha b^2$. Thus we choose

$$c = 4/\alpha b. \tag{2.5}$$

This gives $a \geqq 32b(8/3\alpha b^2 + 1)^2$. Hence, by (2.3), we can take

$$\alpha b^2 = \tfrac{1}{4}\pi^2, \tag{2.6}$$

$$a = 32b(32/3\pi^2 + 1)^2. \tag{2.7}$$

Reversing the steps, given a, we choose b so that (2.7) holds and then α and c so that (2.6) and (2.5) hold. Then (2.2) gives the required $q(x)$.

Comparing our result with (1.4), we have

$$\int_0^a q^2(x)dx = 2\alpha^2 b = 2(\tfrac{1}{4}\pi^2)^2 b^{-3},$$

which, by (2.7) gives (1.5) since $32/3\pi^2 > 1$. No doubt the multiple of a^{-3} which occurs here could be reduced to some extent at the cost of more complicated analysis.

3. We return now to (1.1) with general $q(x)$ satisfying (1.3). If $q(x)$ has the Fourier series $\sum c_r \exp(2\pi rxi/a)$, then the expression (1.2) in Putnam's result is

$$2\left(2\sum_1^\infty |c_r|^2\right)^{1/2}. \tag{3.1}$$

Our extension is as follows:

THEOREM. *For any integer $N \geqq 1$, a gap in S with midpoint μ has length not exceeding*

$$2\left(2\sum_{N+1}^\infty |c_r|^2\right)^{1/2} + \frac{2\pi}{a}\left(2\sum_1^N r^2|c_r|^2 \Big/ \left(\mu - 2\sum_1^N |c_r|\right)\right)^{1/2} \tag{3.2}$$

if $\mu > 2\sum_1^N |c_r|$.

We use the result that a gap in S with midpoint μ has length not exceeding

$$2 \liminf_{m\to\infty} \left\| \{q(x) - \mu\} f_m(x) - f_m''(x) \right\| \tag{3.3}$$

for any sequence $\{f_m\}$ such that $\|f_m\| = 1$, f_m converges weakly to zero in $L^2(-\infty, \infty)$, and $f_m(x)$ has compact support and a continuous second derivative. Here, the norm is that of the complex space $L^2(-\infty, \infty)$ [**3**, inequality (8)]; [**2**, (3.3)]. Let $g(x)$ be any fixed function defined for $0 \leqq x \leqq 1$ such that

$$g(0) = 0, \quad g(1) = 1, \quad g'(0) = g''(0) = g'(1) = g''(1) = 0, \quad 0 \leqq g(x) \leqq 1$$

and let

$$Q(x) = \int_0^x \left(\mu - \sum_{-N}^{N} c_r \exp(2\pi r t i/a) \right)^{1/2} dt. \tag{3.4}$$

Now define $f_m(x) = B_m e^{iQ(x)} h_m(x)$, where

$$\begin{aligned} h_m(x) &= 1 & |x| &\leqq ma - 1, \\ &= g(ma - |x|) & ma - 1 < |x| &\leqq ma, \\ &= 0 & |x| &> ma, \end{aligned}$$

and B_m is the normalization constant. Now $Q(x)$ is real-valued if $\mu > 2\sum_1^N |c_r|$ and it is then easy to verify that

$$|f_m(x)| \leqq B_m, \tag{3.5}$$

$$B_m \sim (2ma)^{-1/2} \qquad (m \to \infty), \tag{3.6}$$

and that $f_m(x)$ satisfies the above conditions for (3.3) (cf. [**2**, §2]). We have

$$\begin{aligned} \{q(x) - \mu\} f_m(x) - f_m''(x) &= \{q(x) - \mu + Q'^2(x)\} f_m - 2iB_m e^{iQ} Q' h_m' - iQ'' f_m - B_m e^{iQ} h_m'' \\ &= u_1(x) + u_2(x) + u_3(x) + u_4(x), \end{aligned}$$

say. It is easy to check that $\|u_2(x)\| \to 0$ and $\|u_4(x)\| \to 0$ as $m \to \infty$. By (3.4) and (3.5),

$$\begin{aligned} \|u_1(x)\| &\leqq B_m \left\{ 2m \int_0^a \left(q(x) - \sum_{-N}^{N} c_r \exp(2\pi r x i/a) \right)^2 dx \right\}^{1/2} \\ &= B_m (2ma)^{1/2} \left(2 \sum_{N+1}^{\infty} |c_r|^2 \right)^{1/2}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \|u_3(x)\| &\leqq B_m\left(2m\int_0^a \{Q''(x)\}^2dx\right)^{1/2} \\ &\leqq B_m(2ma)^{1/2}\frac{1}{2}\left(\mu - 2\sum_1^N |c_r|\right)^{-1/2}\left(2\sum_1^N (2\pi r|c_r|/a)^2\right)^{1/2}. \end{aligned} \tag{3.8}$$

The theorem now follows from (3.3), (3.6), (3.7), and (3.8).

Putnam's proof of (3.1) was in essence the case $N=0$ of the above proof. We note that (3.2) is certainly not greater than (3.1) if μ is large enough. Indeed, (3.2) gives a new proof that the gap with midpoint μ tends to zero in length as $\mu\rightarrow\infty$. To see this, for any $\epsilon>0$, we choose N so that the first term in (3.2) is less than $\frac{1}{2}\epsilon$ and then μ so that the second term is less than $\frac{1}{2}\epsilon$. Thus (3.2) is less than ϵ if $\mu\geqq\mu(\epsilon)$.

References

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