

NONCONTINUABLE SOLUTIONS OF DIFFERENTIAL- DELAY EQUATIONS

JAMES A. YORKE¹

Let $q > 0$ and let $C^q (= C([-q, 0], R^d))$ be the continuous functions $\phi: [-q, 0] \rightarrow R^d$, where $\|\phi\| = \sup_{[-q, 0]} |\phi|$. If x is a continuous function with values in R^d defined on at least $[t-q, t]$, then $x_t \in C^q$ is defined by $x_t(s) = x(t+s)$ for $s \in [-q, 0]$. The purpose of this paper is to determine when a function x can be a noncontinuable solution of one of the differential-delay equations

$$(1) \quad \dot{x}(t) = G(x_t),$$

$$(2) \quad \dot{x}(t) = F(t, x_t).$$

We say x is a *solution on* $[t_0, T)$ of (1) or (2) if $t_0 < T$, $x: [t_0 - q, T) \rightarrow R^d$ is continuous, and x is continuously differentiable on $[t_0, T)$ and (1) or (2) is satisfied for all $t \in [t_0, T)$ (allowing $\dot{x}(t_0)$ to be a right-hand derivative).

THEOREM 1. *Let $t_0 - q < t_0 < T$. Let $x: [t_0 - q, T) \rightarrow R$ be a continuously differentiable function satisfying*

$$(3) \quad \lim_{t \rightarrow T} x(t) \text{ does not exist.}$$

(i) *Then, there exists a continuous $G: C^q \rightarrow R^d$ such that x is a (non-continuable) solution on $[t_0, T)$ of (1).*

(ii) *If $\dot{x}(t)$ exists for all $t \in [t_0, T)$, there exists an $F: R \times C^q \rightarrow R^d$ which is locally Lipschitzian in t and x such that x is a (noncontinuable) solution of (DDE) on $[t_0, T)$.*

The following condition is that $|F|$ is bounded on bounded closed subsets of $R \times C^q$. Write $C^q_\beta = \{\phi \in C^q: \|\phi\| \leq \beta\}$.

$$(4) \quad |F| \text{ is bounded on } [t_0, T] \times C^q_\beta \text{ for each } t_0, T \text{ and each } \beta \geq 0.$$

Condition (4) usually holds for equations that appear in the literature. It is stated here to emphasize that the theory of continuability of solutions can be fairly natural when (4) is assumed: *Let x be a non-continuable solution on $[t_0, T)$ of (1), where $t_0 - q < t_0 < T$ and $F: R \times C^q \rightarrow R^d$ is continuous. If F satisfies (4), then there exists $t_n \rightarrow T$ such*

Received by the editors July 31, 1968.

¹ Partially supported by National Science Foundation Grant GP 7846.

that $|x(t_n)| \rightarrow \infty$ as $n \rightarrow \infty$. This result is easy to prove. (See Theorem 2.4(ii) in [1].)

If the theory is to represent the examples in the literature as faithfully as possible, then (4) should be a standard hypothesis. Unlike the results for ordinary differential equations it is not necessarily true that $|x(t)| \rightarrow \infty$ as $t \rightarrow T$ even if (4) is satisfied. Myškis [3, §3] gave a related example of the form $x(t) = f(t, x(t-t^2))$ for which a solution x approaches the boundary of the region only along a sequence. One purpose of this paper is to show what happens if (4) is *not* assumed.

In particular (3) is satisfied for $t_0 - q < t_0 < 0$ and $T = 0$ by $x(t) = t^{-1}$, so that $|x(t)| \rightarrow \infty$, and by $x(t) = \sin(t^{-1})$, so that $|x(t)|$ is bounded. We now prove Theorem 1.

PROOF. Let $x: [t_0 - q, T] \rightarrow R$ be continuously differentiable and satisfy (3). Then $x(t)$ either has no cluster points (and $|x(t)| \rightarrow \infty$) or $x(t)$ has more than one cluster point as $t \rightarrow T$. In either case we claim

- (5) x_t has no limit points in C^q as $t \rightarrow T$, and
- (6) $S = \{x_t: t_0 \leq t < T\}$ is a closed subset of C^q .

To show (5), suppose it is false, i.e., for some sequence $t_n \rightarrow T$ as $n \rightarrow \infty$ and some $\phi \in C^q$, $x_{t_n} \rightarrow \phi$ as $n \rightarrow \infty$. In particular for each $s \in [-q, 0)$, $x_{t_n}(s) \rightarrow \phi(s)$. But

$$x_{t_n}(s) = x(t_n + s) \rightarrow x(T + s) \quad \text{as } t_n \rightarrow T,$$

so $\phi(s) = x(T + s)$ and $\phi(T - t) \rightarrow \phi(0)$ as $t \rightarrow T$, contradicting (4). To see (6) suppose $x_{t_n} \rightarrow \phi$ as $n \rightarrow \infty$ for some $\{t_n\} \subset [t_0, T)$ and some $\phi \in C^q$, $\phi \notin S$. No subsequence $\{t_{n_i}\}$ can converge to a point $\tau \in [t_0, T)$, since then $x_{t_{n_i}} \rightarrow x_\tau$ and $x_\tau = \phi$, contradicting our assumption $\phi \notin S$. Hence $t_n \rightarrow T$. But by (5) ϕ cannot be a limit point so (6) is proved.

To prove (i) we use

TIETZE EXTENSION THEOREM. *If X is a metric space and $g: S \rightarrow R^d$ is continuous and $S \subset X$ is closed, then there exists a continuous $G: X \rightarrow R^d$ such that $G(\phi) = g(\phi)$ for all $\phi \in S$.*

I now sketch a proof of this version of the Tietze theorem for $d = 1$. (For higher dimensions, apply the one-dimensional result to each of the coordinate functions.) The usual result assumes that $g(S) \subset [a, b]$, a compact interval. Let $d(x, y)$ be the given metric, and let $d(x, S) = \inf_{y \in S} d(x, y)$. The function G may then be defined

$$\begin{aligned} G(x) &= g(x) \quad \text{for } x \in S \\ &= \inf_{y \in S} \{g(y) + d(y, x)/d(S, x)\} - 1 \quad \text{for } x \notin S. \end{aligned}$$

Then $G(X) \subset [a, b]$, and it may be shown that G is continuous. This definition of G is an improvement of the one in Dieudonné [2, p. 85]. When $g(S)$ is not assumed bounded, define $G(x) = g(x)$ for $x \in S$ and

$$G(x) = \inf_{y \in S} \left\{ \sup \left\{ -\frac{1}{d(x, S)}, g(y) + d(y, x)/d(S, x) \right\} \right\} - 1 \text{ for } x \notin S.$$

Then $G(x) \geq -d(x, S)^{-1} - 1$ and G can be shown to be continuous.

PROOF OF (i). For $\phi \in S$ (S as in (6)) the derivative $\dot{\phi}(s)$ exists, so using the left-hand derivative of ϕ at 0 we may define

$$g(\phi) = \dot{\phi}(0) \quad \text{for } \phi \in S.$$

Since $g(x_t) = \dot{x}(t)$, we claim g is continuous on S . If g were not continuous, there would exist $\{\phi_n\}_{n=1,2,\dots} \subset S$, and $\phi \in S$, and $\epsilon > 0$ such that $\phi_n \rightarrow \phi$ and $|g(\phi_n) - g(\phi)| > \epsilon$ for all n . There would exist $\{t_n\} \subset [t_0, T)$ such that $\phi_n = x_{t_n}$. Choosing a subsequence of $\{t_n\}$ if necessary, we may assume $t_n \rightarrow \tau$ for some $\tau \in [t_0, T)$. By (5), $\tau \neq T$. Then $\phi = x_\tau$ and $g(x_{t_n}) = g(\phi_n) \rightarrow g(\phi) = g(x_\tau)$ contradicting our assumption on ϵ , so g must be continuous. Let G be the continuous function given by the Extension Theorem where $X = C^q$ and S is given by (6). Then $G: C^q \rightarrow R$ is continuous and $\dot{x}(t) = G(x_t)$ for $t \in [t_0, T)$, so x is a solution of (2) on $[t_0, T)$. ■

PROOF OF (ii). For $t \in R$ and $\phi \in C^q$, define

$$\begin{aligned} F(t, \phi) &= 0 && \text{for } t \geq T \\ &= \dot{x}(t) [1 - \|x_t - \phi\| (T - t)^{-1}] \\ &&& \text{for } t \in [t_0, T), \quad \|x_t - \phi\| \leq T - t, \\ &= 0 && \text{for } t \in [t_0, T) \text{ and } \|x_t - \phi\| \geq T - t, \\ &= F(t_0, \phi) && \text{for } t \leq t_0. \end{aligned}$$

To see that F is continuous at each $(t, \phi) \in R \times C^q$, let $t_n \rightarrow t$ and $\phi_n \rightarrow \phi$ as $n \rightarrow \infty$. It is clear that $F(t_n, \phi_n) \rightarrow F(t, \phi)$ as $n \rightarrow \infty$ if $t \neq T$. If $t = T$ and $F(t_n, \phi_n) \neq 0$ for $n = 1, 2, \dots$, then $\|x_{t_n} - \phi_n\| < [T - t_n]^{-1} \rightarrow 0$ as $n \rightarrow \infty$. Therefore $x_{t_n} \rightarrow \phi$ as $n \rightarrow \infty$, contradicting (5). Hence $F(t_n, \phi_n) = 0$ for all but finitely many n and F is identically 0 in some neighborhood of (T, ϕ) for each $\phi \in C^q$; that is, F is continuous.

Now assume \dot{x} exists. (It need not be continuous.) Let $W = [t_0, T) \times C^q$ and for $(t, \phi) \in W$, define

$$F_1(t, \phi) = \dot{x}(t) [|T - t| - \|x_t - \phi\|] |T - t|^{-1}.$$

From the definition of F , it suffices to show that F_1 is locally Lipschitzian on W : that is, for each $w \in W$, there exists a neighborhood N of w and $L > 0$ such that

$$|F_1(w_1) - F_1(w_2)| \leq L\|w_1 - w_2\| \quad \text{for } w_1, w_2 \in N$$

where $\|w_1 - w_2\| = \|\phi_1 - \phi_2\| + |t_1 - t_2|$ if $w_1 = (t_1, \phi_1)$ and $w_2 = (t_2, \phi_2)$. Choose $w \in W$ and write $w = (t, \phi)$. Let $F_2(w) = \dot{x}(t)$, $F_3(w) = |T - t|$, $F_4(w) = -\|x_t - \phi\|$, and $F_5(w) = |T - t|^{-1}$. Then w has a neighborhood N on which $|F_i|$, $i = 2, \dots, 5$, are bounded and have Lipschitz constants, respectively, $2|\dot{x}(t)|$, 1 , $\sup_{s \in [-q, 0]} |\dot{x}(t+s)| + 1$, and $2(T-t)^{-2}$. Since the sums of Lipschitz functions (on N) are Lipschitz, and the products of bounded (on N) Lipschitz functions are Lipschitz, $F_1 = F_2(F_3 + F_4)F_5$ is Lipschitz on N , and so F_1 is locally Lipschitz on W . In particular if f_i are Lipschitz functions on N with constant L_i and $|f_i|$ is bounded by B_i , for $i = 1, 2$, then $f_1 f_2$ has Lipschitz constant $L_1 B_2 + L_2 B_1$ on N , and the proof of (ii) is completed. ■

REMARKS. Theorem 1 can be sharpened. Let $x: [t_0 - q, T] \rightarrow R^d$ be continuous. Then x is a noncontinuable solution of (1) on $[t_0, T)$ for some continuous $G: C^q \rightarrow R^d$ if and only if x satisfies (3) and \dot{x} exists and is continuous on $[t_0, T)$ (where $\dot{x}(t_0)$ is the right-hand derivative). Furthermore, F can be chosen locally Lipschitz in t and x if and only if \dot{x} is locally Lipschitz on $[t_0, T)$. The function G cannot be locally Lipschitz if $x_{t_1} = x_{t_2}$ for $t_0 \leq t_1 < t_2 < T$, since the solution of (1) with initial condition x_{t_1} is not unique: x is a solution and there is a periodic solution on (t_1, ∞) (with period $t_2 - t_1$).

Note that in Theorem 1, if $\lim_{t \rightarrow T} x(t)$ exists, x cannot be a solution of (1) or (2) unless $\lim_{t \rightarrow T} \dot{x}(t)$ also exists.

The use of the Tietze theorem to define a function G (as in Theorem 1(i)) is of some interest in itself. We sketch another application. Let Q_L be the set of functions $\gamma: R^d \rightarrow R^d$ which satisfy $|\gamma(x) - \gamma(y)| \leq L|x - y|$. For each $\gamma \in Q_L$, each solution $x(\cdot)$ of

$$(7) \quad \dot{x}(t) = \gamma(x(t)),$$

is defined on all of R .

THEOREM 2. For each $L > 0$ and $q > 0$ there exists a continuous $G: C^q \rightarrow R^d$ such that for every $\gamma \in Q_L$, every solution of (7) is also a solution of (1) for all $t \in R$.

That is, G depends only on L and not on γ . We outline the proof. Fix L . Let S_0 be the set of functions $x: R \rightarrow R^d$ which satisfy (7) for some $\gamma \in Q_L$. Now let $S = \{x_t: x(\cdot) \in S_0 \text{ and } t \in R\} \subset C_q$. It can be shown that S is closed, and in fact if $\{\phi_n\} \subset S$ and $\phi_n \rightarrow \phi$ as $n \rightarrow \infty$, then not only is $\phi \in S$, but $\phi_n(t) \rightarrow \phi(t)$ for all $t \in [-q, 0]$. Hence, the function $\phi \rightarrow \phi(0)$ is continuous, (where $\phi(0)$ is a one-sided derivative).

Define g on S as before, $g(\phi) = \dot{\phi}(0)$. Now g is clearly continuous and can be extended to $G: C^q \rightarrow R^d$. ■

BIBLIOGRAPHY

1. J. A. Yorke, *Asymptotic stability for one-dimensional differential-delay equations*, J. Differential Equations (to appear).
2. J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York, 1960.
3. A. D. Myškis, *General theory of differential equations with retarded argument*, Upsehi Mat. Nauk 4 (1949), no. 5 (33), 99–141; English transl., Amer. Math. Soc. Transl. (1) 4 (1962), 207–267.

INSTITUTE FOR FLUID DYNAMICS AND APPLIED MATHEMATICS,
UNIVERSITY OF MARYLAND