MULTINOMIAL REPRESENTATION OF SOLUTIONS OF A CLASS OF SINGULAR INITIAL VALUE PROBLEMS

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1. Introduction. Let $x=(x_1, \cdots, x_n)$, $x^{2k}=x_1^{2k_1}\cdots x_n^{2k_n}$ where k is the vector $(k_1, \cdots, k_n)$ with k_j a nonnegative integer $(j=1, \cdots, n)$, and let $\|k\|=k_1+\cdots+k_n$. Let $\phi(x)$ be an analytic function of $x_1^2, \cdots, x_n^2$ in a domain D that includes the origin and let $\Delta_j=D_{x_j}^2+(\alpha_j/x_j)D_{x_j}$; $\alpha_j\geqq 0, j=1, \cdots, n$. Finally, let $a>-1$ and $\epsilon_j=1$ if $j=1, \cdots, m$ and $\epsilon_j=-1$ if $j=m+1, \cdots, n$. In this note, we shall be concerned with the question of obtaining representations of analytic solutions of the problem

$$
\begin{aligned}
&\text{(a)}\quad \left(D_t^2+\frac{a}{t}D_t\right)u(x, t) = \sum_{j=1}^{n}\epsilon_j\Delta_j u(x, t),\\
&\text{(b)}\quad u(x, 0) = \phi(x), \qquad u_t(x, 0) = 0
\end{aligned}
\tag{1.1}
$$

in terms of a set $\{P_k(x, t)\}$ of associated multinomials. These multinomials $P_k(x, t)$ are solutions of (1.1) corresponding to the choice $\phi(x)=x^{2k}$ in (1.1b). It will be shown that these $P_k(x, t)$ have the explicit forms

$$
\begin{aligned}
P_k(x, t) &= \Gamma\left(\frac{a+1}{2}\right)\sum_{j=0}^{\|k\|}\frac{t^{2j}}{\Gamma(j+(a+1)/2)}P_{k,j}(x)\\
\text{with } P_{k,j}(x) &= \sum_{r_s, k_s-r_s\geq 0;\, \|r\|=j}\ \prod_{s=1}^{n}\\
&\quad\cdot\left\{\epsilon_s^{r_s}\binom{k_s}{r_s}\cdot\frac{\Gamma(k_s+(\alpha_s+1)/2)}{\Gamma(k_s-r_s+(\alpha_s+1)/2)}x_s^{2(k_s-r_s)}\right\}.
\end{aligned}
\tag{1.2}
$$

Multinomials of this type have been constructed by E. P. Miles and E. C. Young [3] when $m=n$ or $m=0$. In these cases (1.1a) reduces to either the generalized Euler-Poisson-Darboux or the generalized Beltrami equation. R. P. Gilbert and H. C. Howard ([5], [6]) have discussed analyticity properties of solutions of special cases of (1.1). (See [5] for additional reference.)

The authors have examined similar problems when $n=1$ [2]. In

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these cases, the corresponding $P_k(x, t)$ are defined in terms of Jacobi polynomials. The growth bounds and asymptotic estimates for these Jacobi polynomials then permit the obtaining of global regions of convergence from a knowledge of the singularities of the given data function $\phi(x)$. Suitable bounds on the $P_k(x, t)$ for $n \geqq 2$ for determining a convergence region can be obtained by employing the method of related partial differential equations [1]. It will be found that these $P_k(x, t)$, $n \geqq 2$, can be expressed as a convolution of n polynomials $P_{k_j}(x_j, t)$, $j=1, \cdots, n$. Bounds for the $P_k(x, t)$ then follow from the bounds on the simpler polynomials entering this convolution. In particular, we shall prove

THEOREM 1. *Let* $\phi(x) = \sum_{\|k\|=0}^{\infty} a_k x^{2k}$ *be analytic in* $(x_1^2, \cdots, x_n^2)$ *and converge in a domain* D *that includes the origin. Then the series* $\sum_{\|k\|=0}^{\infty} a_k P_k(x, t)$ *converges to an analytic solution of* (1.1) *at least in a region* S *where* S *is defined by* $(x, t) \in S$ *if and only if*

$$(|x_1| + |t|, \cdots, |x_m| + |t|, \quad (x_{m+1}^2 + t^2)^{1/2}, \cdots, (x_n^2 + t^2)^{1/2}) \in D.$$

If $n=1$, *this convergence region is maximal* [2].

The methods used here can be employed for treating a much wider class of initial value problems. For brevity, we omit these cases. However, it should be clear that the key to handing the multi-space variable problem is a thorough treatment of the one space variable problem.

2. **Some background results.** It has been shown in [2] that if $n=1$ and $\phi(x) = x^{2k}$ in (1.1), then the solution of (1.1) with $\alpha_1 = \alpha$ is given by

$$(2.1) \quad u(x, t) = \frac{(\pm 1)^k k! \Gamma((a+1)/2)}{\Gamma(k + (a+1)/2)} (t^2 \mp x^2)^k P_k^{((a-1)/2, (a-1)/2)} \left(\frac{t^2 \pm x^2}{t^2 \mp x^2} \right)$$

where $P_k^{(\mu,\nu)}(y)$ denotes a Jacobi polynomial of degree k in y with parameters μ and ν. The upper sign is used in (2.1) if $\epsilon_1 = 1$ and the lower sign is used if $\epsilon_1 = -1$. Further, we have the following bounds on this $u(x, t)$:

$$(2.2)\ (a) \quad |u(x, t)| \leqq \frac{\Gamma((a+1)/2)\Gamma(k + (\alpha+1)/2)}{2\sqrt{\pi}\Gamma(k + 1/2)} \cdot [(x+t)^{2k} + (x-t)^{2k}], \qquad \epsilon_1 = 1,$$

$$(2.2)\ (b) \quad |u(x, t)| \leqq K \frac{k! \Gamma((a+1)/2)}{\Gamma(k + (a+1)/2)} k^q (x^2 + t^2)^k, \qquad \epsilon_1 = -1$$

where $q=\max((\alpha-1)/2,\ (a-1)/2,\ -1/2)$ and $K>1$. In the bound (2.2a), we have used the fact that $a\geqq 0$. However, if $-1<a<0$, an application of the relation

$$P_k^{(\mu,\nu-1)}(x) = \left\{\frac{\mu+\nu+k}{\mu+\nu+2k}\right\} P_k^{(\mu,\nu)}(x) + \left\{\frac{\mu+k}{\mu+\nu+2k}\right\} P_{k-1}^{(\mu,\nu)}(x)$$

(see [**4**, p. 265]) along with (2.1) and the bound (2.2a) shows that

$$|u(x,t)| \leqq \frac{2k\Gamma((a+1)/2)}{\sqrt{\pi}\Gamma(k-1/2)}\Gamma(k+(\alpha+1)/2)\{|x|+|t|\}^{2k}, \qquad a>-1. \tag{2.3}$$

We now show that the multinomials $P_k(x,t)$ given by (1.2) for $n\geqq 2$ satisfy (1.1) with $\phi(x)=x^{2k}$. It is clear that if $P_k(x,t)$ is a solution of (1.1), then $P_k(x,t)$ must have the form $\sum_{j=0}^{\|k\|} t^{2j}Q_j(x)$. If this is substituted into (1.1), we obtain $Q_0(x)=x^{2k}$ along with the recursion formula

$$Q_j(x) = \frac{1}{2j(2j+a+1)}\{\epsilon_1\Delta_1+\cdots+\epsilon_n\Delta_n\}\cdot Q_{j-1}(x), \qquad j=1,2,\cdots,\|k\|.$$

With this, we readily obtain

$$P_k(x,t) = \Gamma\left(\frac{a+1}{2}\right)\sum_{j=0}^{\|k\|}\frac{t^{2j}}{2^{2j}j!\Gamma(j+(a+1)/2)}\{\epsilon_1\Delta_1+\cdots+\epsilon_n\Delta_n\}^j x^{2k}.$$

The solution form (1.2) follows by expanding the operator and using the fact that

$$\begin{aligned}\Delta_j^{r_j}x_j^{2k_j} &= \frac{2^{2r_j}k_j!\Gamma(k_j+(\alpha_j+1)/2)}{(k_j-r_j+1)!\Gamma(k_j-r_j+(\alpha_j+1)/2)}x_j^{2(k_j-r_j)}, \quad r_j\leqq k_j,\\ &= 0 \qquad \text{if } r_j>k_j.\end{aligned} \tag{2.4}$$

Finally, we recall some elementary results from "heat type" equations and related partial differential equations. Let $v(x,t)$ be a solution of the problem

$$D_t v(x,t) = \sum_{j=1}^{n}\epsilon_j\Delta_j v(x,t), \qquad v(x,0)=x^{2k}. \tag{2.5}$$

Then $v(x, t) = \prod_{j=1}^{n} v_{k_j}(x_j, t)$ where $v_{k_j}(x_j, t)$ satisfies the problem

$$D_t v_{k_j}(x_j, t) = \epsilon_j \Delta_j v_{k_j}(x_j, t), \qquad v_{k_j}(x_j, 0) = x_j^{2k_j}. \tag{2.6}$$

An application of formula (4.2) of [1] (modified slightly to fit the present generalization) shows that the solution of (1.1) with $\phi(x) = x^{2k}$ can be written in the form

$$P_k(x, t) = t^{1-a}\Gamma\left(\frac{a+1}{2}\right)\mathfrak{L}_s^{-1}\left\{\prod_{j=1}^{n}\left[s^{-((a+1)/2n)} v_{k_j}(x_j, 1/4s)\right]\right\}_{s\to t^2}. \tag{2.7}$$

By the convolution theorem for Laplace transforms, as applied to the right member of (2.7), it follows that

$$\begin{aligned} &P_k(x, t) \\ &= \frac{t^{1-a}\Gamma((a+1)/2)}{\{\Gamma((a+1)/2n)\}^n}\left\{\prod_{j=1}^{n} * \left[\tau^{(a+1)/2n-1} W_{k_j}(x_j, \tau^{1/2})\right]\right\}_{\tau=t^2}. \end{aligned} \tag{2.8}$$

We understand this notation to mean that

$$U_i(x_i, \tau^{1/2}) * U_j(x_j, \tau^{1/2}) = \int_0^\tau U_i(x_i, \sigma^{1/2}) U_j(x_j, (\tau - \sigma)^{1/2}) d\sigma$$

and

$$\prod_{j=1}^{n} * U_j(x_j, \tau^{1/2}) = U_1(x_1, \tau^{1/2}) * U_2(x_2, \tau^{1/2}) * \cdots * U_n(x_n, \tau^{1/2}).$$

The $W_{k_j}(x_j, t)$ in (2.8) are solutions of the problems

$$\begin{aligned} &D_t^2 W_{k_j}(x_j, t) + \frac{((a+1)/n - 1)}{t} D_t W_{k_j}(x_j, t) = \epsilon_j \Delta_j W_{k_j}(x_j, t), \\ &W_{k_j}(x_j, 0) = x_j^{2k_j}, \quad D_t W_{k_j}(x_j, 0) = 0, \qquad j = 1, \cdots, n. \end{aligned} \tag{2.9}$$

Since $a+1>0$, the Euler-Poisson-Darboux or Beltrami problems in (2.9) have unique solutions in the class of polynomials. These are given by making the following changes in (2.1). Replace a by $(a+1)/n-1$, k by k_j, and x by x_j, $j=1, 2, \cdots, n$.

3. **Proof of Theorem 1.** An introduction of the bounds (2.2b) and (2.3) into (2.8) now permits us to obtain growth bounds on the $P_k(x, t)$. For example, if $m=1$, $n=2$, and

$$q_2 = \max((\alpha_2 - 1)/2), ((a+1)/4 - 1, -1/2),$$

we obtain

$$
\begin{aligned}
|P_k(x,t)| &\leqq \frac{t^{1-a}\Gamma((a+1)/2)}{\{\Gamma((a+1)/4)\}^2} \\
&\quad \cdot \int_0^\tau \sigma^{(a+1)/4-1}(\tau-\sigma)^{(a+1)/4-1}|W_{k_1}(x_1,\sigma^{1/2})| \cdot |W_{k_2}(x_2,(\tau-\sigma)^{1/2})|\,d\sigma \\
&\leqq t^{1-a}\Gamma\left(\frac{a+1}{2}\right)\frac{2}{\sqrt{\pi}}K\frac{k_1\Gamma(k_1+(\alpha_1+1)/2}{\Gamma(k_1-1/2)}\frac{k_2^{q_2}k_2!}{\Gamma(k_2+(a+1)/2n)} \\
&\quad \cdot\left(\int_0^\tau \sigma^{(a+1)/4-1}(\tau-\sigma)^{(a+1)/4-1}\{|x_1|+|\sigma|\}^{2k_1}\right. \\
&\qquad\qquad \left.\cdot\{x_2^2+(\tau-\sigma)^2\}^{k_2}d\sigma\right)_{\tau=t^2} \\
&\leqq t^{1-a}\Gamma\left(\frac{a+1}{2}\right)\{|x_1|+|t|\}^{2k_1}\{x_2^2+t^2\}^{k_2} \\
&\qquad \cdot\frac{2K}{\sqrt{\pi}}\frac{k_1\Gamma(k_1+(\alpha_1+1)/2)}{\Gamma(k_1-1/2)}\frac{k_2^{q_2}k_2!}{\Gamma(k_2+(a+1)/4)} \\
&\qquad \cdot\left(\int_0^\tau \sigma^{(a+1)/4-1}(\tau-\sigma)^{(a+1)/4-1}d\sigma\right)_{\tau=t^2} \\
&= \left\{\Gamma\left(\frac{a+1}{4}\right)\right\}^2\frac{2K}{\sqrt{\pi}}\frac{k_1\Gamma(k_1+(\alpha_1+1)/2)}{\Gamma(k_1-1/2)}\frac{k_2^{q_2}k_2!}{\Gamma(k_2+(a+1)/4)} \\
&\qquad \cdot\{|x_1|+|t|\}^{2k_1}\{x_2^2+t^2\}^{k_2}.
\end{aligned}
$$

Repeated applications of this argument permit us to prove that, in general,

$$
\begin{aligned}
|P_k(x,t)| &\leqq \left\{\Gamma\left(\frac{a+1}{2n}\right)\right\}^n\cdot\frac{2^mK^{n-m}}{\pi^{m/2}} \\
&\quad \cdot\left\{\prod_{j=1}^m k_j\frac{\Gamma(k_j+(\alpha_j+1)/2)}{\Gamma(k_j-1/2)}(|x_j|+|t|)^{2k_j}\right\} \\
&\quad \cdot\left\{\prod_{j=m+1}^n \frac{k_j^{q_j}k_j!}{\Gamma(k_j+(a+1)/2n)}(x_j^2+t^2)^{k_j}\right\}
\end{aligned}
\tag{3.1}
$$

where $q_j=\max((\alpha_j-1)/2,\ (a+1)/2n-1,\ -1/2)$, $j=m+1,\cdots,n$.

It is clear from the form of the right member of (3.1) that there exist constants C, $p_1,\cdots,p_n$ with $p_j=p_j(\alpha_j)$, $j=1,\cdots,m$ and $p_j=p_j(\alpha_j,a,n)$ for $j=m+1,\cdots,n$ such that

$$(3.2)\ |P_k(x,t)| \leqq C\left\{\prod_{j=1}^{n} k_j^{p_j}\right\}(|x_1|+|t|)^{2k_1}\cdots(|x_m|+|t|)^{2k_m}$$
$$\cdot(x_{m+1}^2+t^2)^{k_{m+1}}\cdots(x_n^2+t^2)^{k_n}.$$

Now, $|\phi(x)| \leqq \sum_{\|k\|=0}^{\infty} |a_k|\ |x_1|^{2k_1}\cdots|x_n|^{2k_n}$, this last series converging for $x\in D$. But for $x\in D$, the series

$$(3.3)\qquad \sum_{\|k\|=0}^{\infty} |a_k| \left\{\prod_{j=1}^{n} k_j^{p_j}\right\} |x_n|^{2k_1}\cdots|x_n|^{2k_n}$$

also converges. If $u(x,t)=\sum_{\|k\|=0}^{\infty} a_k P_k(x,t)$, it follows from (3.2) that

$$|u(x,t)| \leqq \sum_{\|k\|=0}^{\infty} |a_k|\cdot|P_k(x,t)|$$
$$\leqq C\sum_{\|k\|=0}^{\infty} |a_k| \left\{\prod_{j=1}^{n} k_j^{p_j}\right\}\{|x_1|+|t|\}^{2k_1}\cdots\{|x_m|+|t|\}^{2k_m}$$
$$\cdot\{(x_{m+1}^2+t^2)^{1/2}\}^{2k_{m+1}}\cdots\{(x_n^2+t^2)^{1/2}\}^{2k_n}.$$

A comparison of this last series with the series (3.3) shows that this dominating series converges in the region S as stated in Theorem 1. This completes the proof of convergence. What remains to be shown is that the function defined by the series is a solution of the differential equation and satisfies the initial conditions. The method is similar to that given in [2] and will not be repeated here.

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