MULTINOMIAL REPRESENTATION OF SOLUTIONS
OF A CLASS OF SINGULAR INITIAL
VALUE PROBLEMS

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1. Introduction. Let \( x = (x_1, \ldots, x_n), x^a = x_1^{a_1} \cdots x_n^{a_n} \) where \( k \) is the vector \((k_1, \ldots, k_n)\) with \( k_j \) a nonnegative integer \((j = 1, \ldots, n)\), and let \( \|k\| = k_1 + \cdots + k_n \). Let \( \phi(x) \) be an analytic function of \( x_1, \ldots, x_n \) in a domain \( D \) that includes the origin and let \( \Delta_j = D_{x_j}^2 + (\alpha_j/x_j) D_{x_j}; \alpha_j \geq 0; j = 1, \ldots, n \). Finally, let \( a > -1 \) and \( \epsilon_j = 1 \) if \( j = 1, \ldots, m \) and \( \epsilon_j = -1 \) if \( j = m + 1, \ldots, n \). In this note, we shall be concerned with the question of obtaining representations of analytic solutions of the problem

\[
\begin{align*}
\text{(a)} & \quad \left( D_t^2 + \frac{a}{t} D_t \right) u(x, t) = \sum_{j=1}^{n} \epsilon_j \Delta_j u(x, t), \\
\text{(b)} & \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = 0
\end{align*}
\]

(1.1)

in terms of a set \( \{P_k(x, t)\} \) of associated multinomials. These multinomials \( P_k(x, t) \) are solutions of (1.1) corresponding to the choice \( \phi(x) = x^a \) in (1.1b). It will be shown that these \( P_k(x, t) \) have the explicit forms

\[
P_k(x, t) = \Gamma \left( \frac{a+1}{2} \right) \sum_{j=0}^{\|k\|} \frac{t^{\|k\|}}{\Gamma(j + (a + 1)/2)} P_{k_j}(x)
\]

(1.2)

with \( P_{k_j}(x) = \sum_{r_s, k_s - r_s, \sum |r_s| = j} \prod_{s=1}^{n} \left\{ \frac{r_s}{k_s} \right\} \frac{\Gamma(k_s + (\alpha_s + 1)/2)}{\Gamma(k_s - r_s + (\alpha_s + 1)/2)} x^{k_s - r_s} \). Multinomials of this type have been constructed by E. P. Miles and E. C. Young [3] when \( m = n \) or \( m = 0 \). In these cases (1.1a) reduces to either the generalized Euler-Poisson-Darboux or the generalized Beltrami equation. R. P. Gilbert and H. C. Howard ([5], [6]) have discussed analyticity properties of solutions of special cases of (1.1) (see [5] for additional reference).

The authors have examined similar problems when \( n = 1 \) [2]. In

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these cases, the corresponding $P_k(x, t)$ are defined in terms of Jacobi polynomials. The growth bounds and asymptotic estimates for these Jacobi polynomials then permit the obtaining of global regions of convergence from a knowledge of the singularities of the given data function $\phi(x)$. Suitable bounds on the $P_k(x, t)$ for $n \geq 2$ can be obtained by employing the method of related partial differential equations [1]. It will be found that these $P_k(x, t)$, $n \geq 2$, can be expressed as a convolution of $n$ polynomials $P_k(x_j, t)$, $j = 1, \ldots, n$. Bounds for the $P_k(x, t)$ then follow from the bounds on the simpler polynomials entering this convolution. In particular, we shall prove

**Theorem 1.** Let $\phi(x) = \sum_{k=0}^{\infty} a_k x^{2k}$ be analytic in $(x_1, \ldots, x_n)$ and converge in a domain $D$ that includes the origin. Then the series $\sum_{k=0}^{\infty} a_k P_k(x, t)$ converges to an analytic solution of (1.1) at least in a region $S$ where $S$ is defined by $(x, t) \in S$ if and only if

$$
(x_1 + t, \ldots, x_n + t, (x_{n+1} + t)^{1/2}, \ldots, (x_m + t)^{1/2}) \in D.
$$

If $n = 1$, this convergence region is maximal [2].

The methods used here can be employed for treating a much wider class of initial value problems. For brevity, we omit these cases. However, it should be clear that the key to handling the multi-space variable problem is a thorough treatment of the one space variable problem.

2. Some background results. It has been shown in [2] that if $n = 1$ and $\phi(x) = x^{2k}$ in (1.1), then the solution of (1.1) with $\alpha_1 = \alpha$ is given by

$$
u(x, t) = \frac{(-1)^k \Gamma((a + 1)/2)}{2^{(a+1)/2} k! \Gamma(k + (a + 1)/2)} \frac{(t^2 + x^2)^k}{(t^2 + x^2)} P_k^\epsilon(x)
$$

where $P_k^{(\mu, \nu)}(y)$ denotes a Jacobi polynomial of degree $k$ in $y$ with parameters $\mu$ and $\nu$. The upper sign is used in (2.1) if $\epsilon_1 = 1$ and the lower sign is used if $\epsilon_1 = -1$. Further, we have the following bounds on this $u(x, t)$:

$$
|u(x, t)| \leq \frac{\Gamma((a + 1)/2) \Gamma(k + (a + 1)/2)}{2 \sqrt{\pi} \Gamma(k + 1/2)} (x + t)^{2k} + (x - t)^{2k}, \quad \epsilon_1 = 1,
$$

(2.2) (a)

$$
|u(x, t)| \leq K \frac{\Gamma((a + 1)/2) \Gamma(k + (a + 1)/2)}{\Gamma(k + (a + 1)/2)} ^{2k} (x^2 + t^2)^k, \quad \epsilon_1 = -1
$$

(2.2) (b)
where $g = \max((\alpha - 1)/2, (a - 1)/2, -1/2)$ and $K > 1$. In the bound (2.2a), we have used the fact that $a \geq 0$. However, if $-1 < a < 0$, an application of the relation

\[
P_k^{(a,v,k)}(x) = \left\{ \frac{\mu + \nu + k}{\mu + \nu + 2k} \right\} P_k^{(a,v)}(x) + \left\{ \frac{\mu + k}{\mu + \nu + 2k} \right\} P_{k-1}^{(a,v)}(x)
\]

(see [4, p. 265]) along with (2.1) and the bound (2.2a) shows that

\[
\left| u(x, t) \right| \leq \frac{2k\Gamma((a + 1)/2)}{\sqrt{\pi}\Gamma(-1/2)} \Gamma(k + (a + 1)/2) \{ |x| + |t| \}^{2k},
\]

(2.3)

We now show that the multinomials $P_k(x, t)$ given by (1.2) for $n \geq 2$ satisfy (1.1) with $\phi(x) = x^{2k}$. It is clear that if $P_k(x, t)$ is a solution of (1.1), then $P_k(x, t)$ must have the form $\sum_{j=0}^{\|k\|} t^j Q_j(x)$. If this is substituted into (1.1), we obtain $Q_0(x) = x^{2k}$ along with the recursion formula

\[
Q_j(x) = \frac{1}{2j(2j + a + 1)} \left\{ \epsilon_1 \Delta_1 + \cdots + \epsilon_n \Delta_n \right\} Q_{j-1}(x), \quad j = 1, 2, \cdots, \|k\|.
\]

With this, we readily obtain

\[
P_k(x, t) = \Gamma\left( \frac{a + 1}{2} \right) \sum_{j=0}^{\|k\|} \frac{t^{2j}}{2^{2j+1}\Gamma(j + (a + 1)/2)} \left\{ \epsilon_1 \Delta_1 + \cdots + \epsilon_n \Delta_n \right\} x^{2k}.
\]

The solution form (1.2) follows by expanding the operator and using the fact that

\[
\Delta_j x_j^{2k} = \frac{2r_j k_j! \Gamma(k_j + (\alpha_j + 1)/2)}{(k_j - r_j + 1)! \Gamma(k_j - r_j + (\alpha_j + 1)/2)} x^{2(k_j-r_j)}, \quad r_j \leq k_j,
\]

(2.4)

= 0 \quad \text{if} \quad r_j > k_j.

Finally, we recall some elementary results from "heat type" equations and related partial differential equations. Let $v(x, t)$ be a solution of the problem

\[
D_p(x, t) = \sum_{j=1}^{n} \epsilon_j \Delta_j v(x, t), \quad v(x, 0) = x^{2k}.
\]

(2.5)
Then $v(x, t) = \prod_{j=1}^{n} v_{k_j}(x_j, t)$ where $v_{k_j}(x_j, t)$ satisfies the problem

$$D_i v_{k_j}(x_j, t) = \epsilon_j \Delta v_{k_j}(x_j, t), \quad v_{k_j}(x_j, 0) = x_j^{2k_j}.$$  

An application of formula (4.2) of [1] (modified slightly to fit the present generalization) shows that the solution of (1.1) with $\phi(x) = x^{2k}$ can be written in the form

$$P_k(x, t) = t^{1-a} \frac{\Gamma((a + 1)/2)}{\Gamma((a + 1)/2n)} \prod_{j=1}^{n} \left\{ \int_{s}^{\infty} \frac{e^{-(a+1)/2n} v_{k_j}(x_j, s)}{s^{(a+1)/2n-1}} ds \right\}.$$  

By the convolution theorem for Laplace transforms, as applied to the right member of (2.7), it follows that

$$P_k(x, t) = t^{1-a} \frac{\Gamma((a + 1)/2)}{\Gamma((a + 1)/2n)} \prod_{j=1}^{n} \left\{ \int_{s}^{\infty} \frac{e^{-(a+1)/2n} W_{k_j}(x_j, s^{1/2})}{s^{(a+1)/2n-1}} ds \right\}.$$

We understand this notation to mean that

$$U_i(x_i, \tau^{1/2}) * U_j(x_j, \tau^{1/2}) = \int_{0}^{\tau} U_i(x_i, \sigma^{1/2}) U_j(x_j, (\tau - \sigma)^{1/2}) d\sigma$$

and

$$\prod_{j=1}^{n} U_j(x_j, \tau^{1/2}) = U_1(x_1, \tau^{1/2}) * U_2(x_2, \tau^{1/2}) * \cdots * U_n(x_n, \tau^{1/2}).$$

The $W_{k_j}(x_j, t)$ in (2.8) are solutions of the problems

$$D_i W_{k_j}(x_j, t) + \frac{(a + 1)/n - 1}{t} D_t W_{k_j}(x_j, t) = \epsilon_j \Delta_j W_{k_j}(x_j, t),$$

$$W_{k_j}(x_j, 0) = x_j^{2k_j}, \quad D_t W_{k_j}(x_j, 0) = 0, \quad j = 1, \ldots, n.$$  

Since $a+1 > 0$, the Euler-Poisson-Darboux or Beltrami problems in (2.9) have unique solutions in the class of polynomials. These are given by making the following changes in (2.1). Replace $a$ by $(a+1)/n-1$, $k$ by $k_j$, and $x$ by $x_j$, $j=1, 2, \ldots, n$.

3. **Proof of Theorem 1.** An introduction of the bounds (2.2b) and (2.3) into (2.8) now permits us to obtain growth bounds on the $P_k(x, t)$. For example, if $m=1$, $n=2$, and

$$q_2 = \max((\alpha_2 - 1)/2), ((a + 1)/4 - 1, -1/2),$$

we obtain
\[
| P_k(x, t) | \leq \frac{t^{l-m} \Gamma((a+1)/2)}{\Gamma((a+1)/4)}^2 \\
\cdot \int_0^\tau \sigma^{(a+1)/4-1}(\tau - \sigma)^{(a+1)/4-1} \left| W_{k_1}(x_1, \sigma^{1/2}) \right| \cdot \left| W_{k_2}(x_2, (\tau - \sigma)^{1/2}) \right| d\sigma \\
\leq \frac{t^{l-m} \Gamma((a+1)/2)}{\Gamma((a+1)/4)}^2 \frac{2K}{\sqrt{\pi}} \frac{k_1 \Gamma(k_1 + (a+1)/2)}{\Gamma(k_1 - 1/2)} \frac{k_2 \gamma_2 k_2!}{\Gamma(k_2 + (a+1)/4)} \\
\cdot \left( \int_0^\tau \sigma^{(a+1)/4-1}(\tau - \sigma)^{(a+1)/4-1} \left| x_1 \right| + \left| \sigma \right| \right)^{2k_1} \\
\cdot \left( x_2^2 + (\tau - \sigma)^2 \right)^{k_2} \\
\cdot \left( \int_0^\tau \sigma^{(a+1)/4-1}(\tau - \sigma)^{(a+1)/4-1} d\sigma \right)
\]

Repeated applications of this argument permit us to prove that, in general,

\[
| P_k(x, t) | \leq \left\{ \Gamma\left( \frac{a+1}{4} \right) \right\}^2 \frac{2K}{\sqrt{\pi}} \frac{k_1 \Gamma(k_1 + (a+1)/2)}{\Gamma(k_1 - 1/2)} \frac{k_2 \gamma_2 k_2!}{\Gamma(k_2 + (a+1)/4)} \\
\cdot \left( \int_0^\tau \sigma^{(a+1)/4-1}(\tau - \sigma)^{(a+1)/4-1} d\sigma \right)
\]

where \( q_j = \max((\alpha_j - 1)/2, (a+1)/2n - 1, -1/2) \), \( j = m+1, \ldots, n \).

It is clear from the form of the right member of (3.1) that there exist constants \( C, \rho_1, \ldots, \rho_n \) with \( \rho_j = \rho_j(\alpha_j) \), \( j = 1, \ldots, m \) and \( \rho_j = \rho_j(\alpha_j, a, n) \) for \( j = m+1, \ldots, n \) such that
\[ (3.2) \left| P_k(x, t) \right| \leq C \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} \left( |x_1| + |t| \right)^{2k_1} \cdots \left( |x_m| + |t| \right)^{2k_m} \]
\[
\cdot \left( |x_{m+1}| + t \right)^{2k_{m+1}} \cdots \left( |x_n| + t \right)^{2k_n}.
\]

Now, \( \phi(x) \leq \sum_{|k| = 0}^{\infty} |a_k| \left| x_1 \right|^{2k_1} \cdots \left| x_n \right|^{2k_n} \), this last series converging for \( x \in D \). But for \( x \in D \), the series
\[ (3.3) \sum_{|k| = 0}^{\infty} \left| a_k \right| \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} \left| x_1 \right|^{2k_1} \cdots \left| x_n \right|^{2k_n} \]
also converges. If \( u(x, t) = \sum_{|k| = 0}^{\infty} a_k P_k(x, t) \), it follows from (3.2) that
\[
\left| u(x, t) \right| \leq \sum_{|k| = 0}^{\infty} \left| a_k \right| \left| P_k(x, t) \right| \leq C \sum_{|k| = 0}^{\infty} \left| a_k \right| \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} \left( |x_1| + |t| \right)^{2k_1} \cdots \left( |x_m| + |t| \right)^{2k_m} \]
\[
\cdot \left( |x_{m+1}| + t \right)^{2k_{m+1}} \cdots \left( |x_n| + t \right)^{2k_n}.\]

A comparison of this last series with the series (3.3) shows that this dominating series converges in the region \( S \) as stated in Theorem 1. This completes the proof of convergence. What remains to be shown is that the function defined by the series is a solution of the differential equation and satisfies the initial conditions. The method is similar to that given in [2] and will not be repeated here.

References


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