A CHARACTERIZATION OF CERTAIN CONFORMALLY EUCLIDEAN SPACES OF CLASS ONE

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1. In this paper we will examine the metrics of conformally Euclidean spaces $C_n (n \geq 4)$ having the following two properties:

(1) They are locally and isometrically imbeddable in Euclidean space of one higher dimension ($\mathbb{E}_{n+1}$), i.e. they are of class one.

(2) With respect to a conformal coordinate system, the matrix of the second fundamental tensor $[\theta_{ij}]$ has diagonal form.

The condition for class one is that there exist a (second fundamental) tensor $[\theta_{ij}]$, satisfying the Gauss (1.1) and Codazzi (1.2) equations:

\[(1.1) \quad R_{hijk} = b_{hi}b_{jk} - b_{hk}b_{ij},\]
\[(1.2) \quad b_{ij,k} = b_{ik,j}.\]

To satisfy (2), we will therefore look for a solution of these equations for which $b_{ij} = 0$ when $i \neq j$.

Sen, in a series of papers ([4], [5] and [6]), has investigated certain conditions for a $C_n$ to be of class one, and obtained [6, Theorem 3] a canonical form for the metric of such a space. His result, however, is incorrect in its full generality (see [3] for a disproof). In 1962, at the meeting of the International Congress of Mathematicians in Stockholm [8], R. Blum presented, without proof, a canonical form for the metric of a $C_n$ satisfying (1) and (2) above and such that $n \geq 4$. In his theorem, however, Blum overlooked an exception, and it is therefore not correct as stated. It is the purpose of this paper to give a proof and a simplification of the corrected result.

Thomas [7] showed that when $\tau$, the rank of the matrix $[\theta_{ij}]$, is greater than or equal to four, equations (1.2) follow as a consequence of equations (1.1). It is therefore logical to consider separately the cases $n \geq 4$ and $n = 3$ (the case $n = 2$ is not considered here; the surfaces $\mathcal{C}_2$ are called isothermal surfaces and form a separate area of study in themselves). It will turn out that for $n \geq 4$, $\tau$ is greater than or equal to four except in two particular cases. In both, however, it is easily verified that equations (1.2) are satisfied because of (1.1). For $n = 3$, the situation is somewhat different and the Codazzi equations must be considered separately as a set of independent conditions. This case will be the object of investigation in a later paper.

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As regards notation, a $C^n$ having property (1) will be denoted $C^n_1$, following Sen's example, while a $C^n$ which has both properties (1) and (2) will be denoted $C^n_2$. Tensor notation used throughout will be essentially that to be found in Eisenhart [2].

2. Referred to a conformal coordinate system, the metric of a $C^n$ is

\[ ds^2 = e^{2\sigma} \sum_i (dx^i)^2, \]

where $\sigma = \sigma(x^1, x^2, \cdots, x^n)$ and $e^{-2\sigma} \neq 0$.

By routine calculation we then obtain the following expression for the Riemann Curvature Tensor:

\[ R_{hijk} = e^{2\sigma} \left[ \delta_{hk}\sigma_{ij} + \delta_{ij}\sigma_{hk} - \delta_{ij}\sigma_{hk} - \delta_{ik}\sigma_{hj} + \sum_m \sigma_{,m}(\delta_{hk}\delta_{ij} - \delta_{ij}\delta_{hk}) \right] \]

where $\sigma_{ij} = \sigma_{,ij} + \sigma_{,i}\sigma_{,j}$, and $\sigma_{,ij}$ (the second covariant derivative of $\sigma$), is given by

\[ \sigma_{,ij} = \partial_i \partial_j \sigma - 2\sigma_{,i}\sigma_{,j} + \delta_{ij} \sum_m \sigma_{,m}. \]

Substituting this expression into (1.1) and considering components yields the following two equations:

\[ \sigma_{ij} = 0 \quad (i \neq j; j = 1, \cdots, n), \]

and

\[ b_{hbb_{ii}} = e^{2\sigma} \left( \sum_m \sigma_{,m} - \sigma_{hh} - \sigma_{ii} \right) \quad (h \neq i; h, i = 1, \cdots, n). \]

We will now consider each of these relations in turn.

3. Equation (2.2) simplifies to

\[ \partial_i \partial_j \sigma - \partial_j \sigma_{,i} = 0 \quad (i \neq j). \]

If we now multiply this by $e^{-\sigma}$ we obtain

\[ \partial_i \partial_j e^{-\sigma} = 0 \quad (i \neq j). \]

Thus

\[ \partial_j e^{-\sigma} = F(x^j) \quad (j = 1, 2, \cdots, n), \]

and hence

\[ e^{-\sigma} = \sum_m f_m \]

where $f_m$ is a function of $x^m$ only.
4. Utilizing equation (3.1), (2.3) reduces to

\[(4.1) \quad b_{hh}b_{kk} = e^{e^2}[f_i' + f'_j] - A] \quad (h \neq k),\]

where

\[(4.2) \quad A = \sum f_m^2.\]

Similarly

\[(4.3) \quad b_{ii}b_{jj} = e^{e^2}[f_i' + f'_j] - A] \quad (i \neq j).\]

Multiplying (4.1) and (4.3) we obtain

\[(4.4) \quad b_{hh}b_{kk}b_{ii}b_{jj} = e^{e^2}(f'_h + f'_i)(f'_k + f'_j)
- A e^{e^2}(f'_h + f'_i + f'_k + f'_j) + A^2]
\quad (h \neq k, i \neq j),\]

and similarly:

\[(4.5) \quad b_{hh}b_{ii}b_{kk}b_{jj} = e^{e^2}(f'_h + f'_i)(f'_k + f'_j)
- A e^{e^2}(f'_h + f'_i + f'_k + f'_j) + A^2]
\quad (h \neq i, k \neq j).\]

Equate (4.4) and (4.5) and simplify. Then

\[(f'_h - f'_i)(f'_k - f'_j) = 0 \quad (i \neq h, i \neq j, k \neq j, k \neq h).\]

From this expression, we then deduce the result that \(f'_i = 2a\) (constant) for all \(i\) except one value, say \(i = 1\). Thus

\[(4.6) \quad f_i = ax^i + bx^i + c_i \quad (i = 2, 3, \ldots, n)\]

while \(f_1\) is arbitrary.

Putting \(f = f_1 + \sum c_i\) and substituting (4.6) and (3.1) into (2.1) we thus obtain

\[(4.7) \quad ds^2 = \frac{\sum (dx^i)^2}{[f(x^1) + a \sum (x^m)^2 + \sum b_m x^m]^2}.\]

It is then a fairly straightforward matter to obtain explicit expressions for the \(b_{ii}\) from equation (4.3), viz

\[(4.8) \quad b_{ii} = e^{e^2}(4af - f'^2 - \sum b_m^2)^{1/2} \quad (i = 2, 3, \ldots, n),\]
and

\begin{align*}
 b_{11} &= b_{tt} + \frac{e^{2\sigma}(f'' - 2a)}{b_{tt}} \quad \text{if } b_{tt} \neq 0 \quad (i \neq 1), \\
 &= 0 \quad \text{if } b_{tt} = 0 \quad (i \neq 1).
\end{align*}

(4.9)

However, there is an interesting exception which arises when \( b_{tt} = 0 \quad (i = 2, \ldots, n) \) and \( a \neq 0 \). If we equate equation (4.8) to zero and solve, we obtain the following two independent solutions:

\begin{align*}
 (1) \quad f(x) &= ax^2 + bx + 2^{-1} b_m/4a, \\
 (2) \quad f(x) &= 2^{-1} b_m/4a.
\end{align*}

Solution (1) implies that \( C_n^1 \) is a Euclidean space and hence \( b_{11} = 0 \) also, as indicated in equation (4.9).

Solution (2), however, yields a contradiction to the effective Gauss equations (4.1), and hence represents a space which is not a \( C_n^1 \). We may see this by direct substitution into equations (4.1):

\begin{align*}
 h, i \neq 1 \quad \text{yields} \quad 4ae^{-\sigma} - A = 0, \\
 h = 1, i \neq 1 \quad \text{yields} \quad 2ae^{-\sigma} - A = 0,
\end{align*}

and together these imply \( e^{-\sigma} = 0 \), i.e. a contradiction.

Furthermore, the space \( C_n \), whose metric is given by equation (4.7) with \( f(x) = \sum_{m=2} b_m/4a \), is not even of class one, i.e. a \( C_n^1 \). This may be seen by obtaining the components of the curvature tensor from this metric and looking for a solution \( [b_{ij}] \), not necessarily diagonal, to equations (1.1). A contradiction is thereby obtained. The space is in fact of class two (see [1]).

Thus to equation (4.7) we must add the condition that \( f(x^1) \neq \sum_{m=2} b_m/4a \).

5. The Codazzi equations follow because of Thomas' result, except for the following situations:

(a) \( b_{11} = b_{ii} = 0 \quad (i = 2, \ldots, n) \), in which case they are satisfied identically.

(b) \( b_{11} = 0, b_{ii} \neq 0 \quad (i = 2, \ldots, n) \), and \( n = 4 \). This situation occurs when \( b_i^2 = e^{2\sigma}(2a - f'') \quad (i = 2, 3, 4) \quad (f'' \neq 2a) \), and \( f \) satisfies the differential equation

\[ f(f'' + 2a) - f' + (f'' - 2a) \sum_{m=2}^4 (ax^m + b_m x^m) - \sum_{m=2}^4 b_m = 0. \]
It can be fairly readily verified that here again the Codazzi equations are satisfied (by converting these equations to the simpler form

\[ \partial_k b_{ij} = \sigma, k (b_{ij} + b_{kj}) \quad (i \neq k) \]

and checking all the cases).

6. Conversely, if we are given a \( C_n \) with metric (4.7), and \( f(x^1) \neq \sum_{m=2}^{n} b_m^2 / 4a \), we may construct a tensor \( [b_{ij}] \) using equations (4.8) and (4.9) such that \( b_{ij} = 0 \) for \( i \neq j \). These in turn satisfy the Gauss and Codazzi equations. Furthermore, the tensor \( [b_{ij}] \) is unique except for sign provided that rank \( [b_{ij}] (=r) \geq 3 \) (see [7, p. 188]). This is always true unless \( C_n \) is a Euclidean space (in which case it is of class zero anyway).

7. We can further simplify the metric (4.7) by considering separately the cases when \( a = 0 \) and \( a \neq 0 \).

\( a \neq 0 \). The transformation \( y^1 = ax^1, y^m = ax^m + b_m / 2 (m = 2, 3, \ldots, n) \) changes the metric to the simpler form

\[ ds^2 = \frac{\sum_{i=1}^{n} (dy^i)^2}{[F(y^1) + \theta]^2} \text{ where } \theta = \sum_{i=2}^{n} (y^i)^2 \text{ and } F(y^1) \neq 0. \]

\( a = 0 \). Here if \( b_m (m = 2, 3, \ldots, n) \) are all zero, we obtain the metric

\[ ds^2 = \frac{\sum_{i=1}^{n} (dx^i)^2}{[f(x^1)]^2}, \]

whereas if the \( b_m \) are not all zero, we may make any orthogonal transformation such that

\[ y^1 = x^1, \quad y^2 = \sum_{m=2}^{n} \frac{b_m}{B} x^m, \text{ where } B = (b_2^2 + b_3^2 + \cdots + b_n^2)^{1/2}, \]

and obtain the metric

\[ ds^2 = \frac{\sum_{i=1}^{n} (dy^i)^2}{[f(y^1) + By^2]^2}. \]

Thus in both cases when \( a = 0 \), the metric of a \( \mathcal{C}_n \) reduces to the form
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\[ ds^2 = \sum_{i=1}^{n} (dx^i)^2 \]

where \( K \) is an arbitrary constant.

8. The following theorem summarizes the results obtained in the preceding sections:

**Theorem.** Let \( C_n^1 (n \geq 4) \) be a conformally Euclidean space of class one, such that, with respect to a conformal coordinate system \( x^1, x^2, \ldots , x^n \), the second fundamental tensor has diagonal form. Then the metric of \( C_n^1 \) takes one of the following two distinct canonical forms:

\[
(I) \quad ds^2 = \sum_{i=1}^{n} (dx^i)^2 \quad \text{where} \quad \theta = \sum_{i=2}^{n} (x^i)^2, \\
\]

\[
(II) \quad ds^2 = \sum_{i=1}^{n} (dx^i)^2 \quad \text{where} \quad \theta = \sum_{i=2}^{n} (x^i)^2, \\
\]

where \( f \) and \( g \) are arbitrary twice differentiable functions of \( x^1 \) only, except that \( f(x^1) \neq 0 \), and \( K \) is an arbitrary constant.

Conversely, if a \( C_n (n \geq 4) \) possesses either of the metrics (I) or (II), then it is a \( C_n^1 \).

**References**


5. ———, *Conformally Euclidean space of class one*, Indian J. Math. 6 (1964), 93-103.


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