EIGENFUNCTION EXPANSIONS OF ANALYTIC FUNCTIONS

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In [5, Theorem 10.2], there was derived a simple result characterizing C^∞ sections f of a vector bundle over a compact manifold, in terms of the rate of decay of the coefficients of f in eigenfunctions of a C^∞ differential operator. Here we derive a similar result for analytic sections, mentioned in [5]. Following the proof are several applications (the first of which motivates the general proof) and an alternate proof based on a conversation with F. E. Browder.

THEOREM. *Let E be a complex vector bundle over the compact real-analytic manifold X. Suppose X has an analytic volume element, that E has an analytic Hermitian inner product, and that A is an analytic, elliptic, normal differential operator of order m on the sections of E. Let $\{\phi_k\}$ and $\{\lambda_k\}$ be respectively the eigensections and eigenvalues of A: $A\phi_k=\lambda_k\phi_k$, and let μ_k be the positive mth root of $|\lambda_k|$. Then $f=\sum f_k\phi_k$ is analytic if and only if the sequence $\{s^{\mu_k}|f_k|\}$ is bounded for some $s>1$.*

The condition of the theorem is equivalent to: $\sum s^{\mu_k}|f_k|^2<\infty$ for some $s>1$, as the proof shows.

By normality of A we mean $A^*A=AA^*$. This guarantees the existence of a basis of orthonormal eigensections, as follows. The null space of A is finite dimensional [5, Theorem 8.3], and if P is orthogonal projection onto this null space, then $P+A$ is normal and has trivial null space and closed range. It follows that $P+A$ is an isomorphism from $H^m(E)$ (the space of sections of E all of whose derivatives of order less than $m+1$ are square integrable) onto $H^0(E)$, the space of square integrable sections of E. Then $P+A$ has an inverse B which is a compact normal operator on $H^0(E)$. Since B has orthonormal eigensections $\{\phi_k\}$ with eigenvalues converging to zero, the eigenvalues $\{\lambda_k\}$ of A converge to infinity. More precisely we have

$$\sum |\lambda_k|^{-2n} < \infty, \tag{1}$$

where n is the dimension of X. For $|\lambda_k|^{2n}$ are the eigenvalues of

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$(AA^*)^n = A^n(A^*)^n$, while $[P+(AA^*)^n]^{-1}$ is an operator of trace class [**5**, Lemma 10.1].

Since the ϕ_k are eigensections of A^*A+I with eigenvalues $|\lambda_k|^2+1$, we may assume that A is positive, that $\lambda_k>0$, and that the order m of A is even.

The proof depends on imbedding X in the open manifold $X' = X\times I$, where I is the open interval (0, 2), and X is identified with $X\times\{1\}$. We rely on the Cauchy-Kowalewski theorem to derive the rate of decay of the coefficients from the analyticity of f, and on the analyticity of solutions of elliptic equations for the converse proof.

Extend the bundle E in the obvious way to X', denoting the extension by $E'=E\times I$. If π is the projection of E onto X, then $\pi': E'\to X'$ is defined by $\pi'(e, t)=(\pi(e), t)$. We consider sections of E' as maps $f': X\times I\to E$ such that $\pi f'(x, t)=x$. Define the operator A' on sections of E' by $A'f'(x, t)=(Af)(x, t)+i(t\partial/\partial t)^m f(x, t)$, where m is the order of A. Then A' is an analytic differential operator on sections of E', and since we have assumed m is even and the characteristic polynomial (symbol) of A is positive definite, it follows that A' is elliptic.

Suppose now f is an analytic section of E. Then for some $\epsilon>0$, there is an analytic solution f' in $X\times[1, 1+\epsilon)\subset X'$ of the Cauchy problem: $A'f'=0$, $f'(x, 1)=f(x)$, $(t\partial/\partial t)^j f'(x, 1)=0$ for $j=1, \cdots, m-1$. Writing $a_k(t)=\int_X\langle f'(x, t), \phi_k(x)\rangle dx$, where $\langle\ ,\ \rangle$ denotes the Hermitian inner product in any fibre of E, we have

$$(t\partial/\partial t)^m a_k(t) = i\int_X\langle Af'(x, t), \phi_k(x)\rangle dx = i\lambda_k a_k(t).$$

Thus $a_k(t)=\sum_{j=1}^m A_{k,j}t^{\theta_j\mu_k}$, where $\mu_k>0$, $(\mu_k)^m=\lambda_k$, and $\{\theta_j\}_1^m$ are the roots of $\theta^m=i=\sqrt{(-1)}$. Applying the data for $t=1$, we find

$$\sum_{j=1}^m A_{k,j}(\theta_j)^p = \delta_{\text{op}}f_k,$$

where $f_k=\int_X\langle f(x), \phi_k(x)\rangle dx$. Thus $A_{k,j}=c_jf_k$, where $\{c_j\}$ is the unique solution of

$$\sum_{j=1}^m c_j(\theta_j)^p = \delta_{\text{op}}, \qquad p = 0, \cdots, m-1. \tag{2}$$

Note that the c_j's are quotients of nonvanishing van der Monde determinants and thus no $c_j=0$. Now

$$\int_X\langle f'(x, t), f'(x, t)\rangle dx = \sum_{k=1}^\infty\left|\sum_{j=1}^m c_jf_kt^{\theta_j\mu_k}\right|^2 < \infty$$

for $1 \leqq t < 1+\epsilon$, so $\{ | \sum_1^m c_j t^{\theta_j \mu_k} | \, |f_k| \}$ is bounded for some fixed $t > 1$. Letting $\theta_1 = e^{i\pi/2m}$, we have $\mathrm{Re}(\theta_1) > \mathrm{Re}(\theta_j)$ for $j = 2, \cdots, m$. Since

$$\sum_1^m c_j t^{\theta_j \mu_k} = t^{\theta_1 \mu_k} \left(c_1 + \sum_2^m c_j t^{(\theta_j - \theta_1)\mu_k} \right),$$

while $\mathrm{Re}(\mu_k(\theta_j - \theta_1)) \to -\infty$ and $c_1 \neq 0$, we have that $\{ | t^{\theta_1 \mu_k} f_k | \}$ is bounded. If α is the real part of θ_1, and $s = t^\alpha$, we then have $s > 1$ and $\{ s^{\mu_k} |f_k| \}$ is bounded, which proves the first part of the theorem.

For the converse, we construct an L^2 solution u of $A'u = 0$ with $u(x, 1) = f(x)$, and then observe that since A' is analytic and elliptic, u is analytic [**2**, §5]. The construction of u proceeds as follows.

First, from the boundedness of $\{ s^{\mu_k} |f_k| \}$ we conclude that $\sum t^{2\mu_k} |f_k|^2 < \infty$ for $0 \leqq t < s$. For if $r = t/s$, $\sum t^{2\mu_k} |f_k|^2 \leqq M \sum r^{2\mu_k}$. Since $\sum (\mu_k)^{-p} < \infty$ for an appropriate p (by (1)), $\sum | \log r^{2\mu_k} |^{-p} < \infty$, and the comparison test shows that $\sum r^{2\mu_k} < \infty$.

Thus writing $u(x, t) = \sum_1^\infty \sum_1^m f_k c_j t^{\theta_j \mu_k} \phi_k(x)$ for $s^{-1} < t < s$ (with c_j as in (2) and $(\theta_j)^m = i$), we have that u is square integrable on every compact subset of $X \times \{ s^{-1} < t < s \}$. It is also easy to show that for each C^∞ section ψ of E' with compact support in $X \times \{ s^{-1} < t < s \}$, we have $(u, (A')^* \phi) = 0$, so that u is a "weak" solution of $A'u = 0$. It follows from standard regularity theorems that u is C^∞ [**1**, Theorem 8.1], and then analytic [**2**, §5]. Finally, since f is the restriction of u to $X \times \{1\}$, f is analytic.

Applications. If we let A be the Laplace operator on the unit sphere $\{ |x| = 1 \}$ in $\mathbf{R}^{n+1}$, then the eigenfunction expansion in question is the spherical harmonic expansion $f(x) = \sum f_{jk} S_{jk}(x)$ $(|x| = 1)$ where S_{jk} is a spherical harmonic of degree j. The eigenvalues are λ_{jk} $-j(j+n-2)$, and k runs from 1 to $(2j+n-2)(j+n-3)!/j!(n-2)!$. Thus it follows easily from the general theorem above that f is analytic if and only if $\sum f_{jk} r^j S_{jk}$ converges (in L^2) for some $r > 1$. Let now $\mathcal{H}$ be the space of functions harmonic in $\{ |x| < 1 \}$, with the topology of uniform convergence on compact sets; and let $\mathcal{A}$ be the set of functions analytic on $\{ |x| = 1 \}$, untopologized. Then we can show immediately that $\mathcal{A}$ *is the dual of* $\mathcal{H}$. For this, use the base of neighborhoods of zero in $\mathcal{H}$ given by

$$U_{r,\delta} = \left\{ u \text{ in } \mathcal{H}\colon \int_{|x|=1} | u(rx) |^2 dx < \delta \right\} \quad \text{for } 0 < r < 1, \ \delta > 0.$$

Suppose $f^\wedge$ is in the dual of $\mathcal{H}$, let $H_{jk}(x) = |x|^j S_{jk}(x/|x|)$, and set

$f_{jk} = f^{\wedge}(H_{jk})$. Suppose $|f^{\wedge}(u)| < 1$ if $u \in U_{r,\delta}$, and let $u = \sum u_{jk} H_{jk}$. Then $|\sum f_{jk} u_{jk}| < 1$ if $\sum |u_{jk}|^2 r^{2j} < \delta$, so $\sum |f_{jk} r^{-j}|^2 < \delta^{-1}$, which shows that the f_{jk} are the spherical harmonic coefficients of a function f in $\mathcal{A}$. Conversely, each function in $\mathcal{A}$ leads to a functional on $\mathcal{H}$, and the isomorphism is established. The same isomorphism can also be realized as follows. Given f analytic on $\{|x| = 1\}$, solve the problem (i) $\Delta v(x) = 0$ in $|x| > 1$, (ii) v bounded in $|x| > 1$, (iii) $v(x) = f(x)$ for $|x| = 1$. Then v extends analytically to $|x| \geqq r$ for some $r < 1$, and for any u in $\mathcal{H}$ we have $f^{\wedge}(u) = \int_{|x|=1} u(rx) v(rx)$. $\mathcal{A}$ can now be given the various topologies of the dual of $\mathcal{H}$. (For a more general result of this type, see Lions and Magenes [7].)

For a second application, suppose f is analytic in $\boldsymbol{R}^{n+1} - \{0\}$, and for some complex λ, $f(tx) = t^{\lambda} f(x)$ for all $t > 0$. Then (except for certain integer values of λ), f defines a unique tempered distribution on $\boldsymbol{R}^{n+1}$, which has a fourier transform $f^{\wedge}$. If $f(x) = |x|^{\lambda} \sum f_{jk} S_{jk}(x/|x|)$, then $f^{\wedge}(x)$ comes from the function $|x|^{-\lambda-n-1} \cdot \sum f_{jk} \gamma_j S_{jk}(x/|x|)$, with $\gamma_j = \pi^{n/2}(-i)^j 2^{\lambda+n} \Gamma((j+n+\lambda)/2)/\Gamma((j-\lambda)/2)$ (see [4]). Since $\sum |f_{jk}|^2 t^{2j} < \infty$ for some $t > 1$, so is $\sum |f_{jk}|^2 |\gamma_j|^2 s^{2j} < \infty$ for some $s > 1$, and f is analytic. The same result holds, with minor rephrasings, for the exceptional integer values mentioned above.

Another corollary of the expansion theorem is the following: *if B is any bounded operator on $H^0(E)$ and $AB = BA$, then B maps analytic functions into analytic functions.* For if $\{\lambda_j\}$ are the *distinct* eigenvalues of A, and S_j is the eigenspace of λ_j, then any f in $H^0(E)$ has the expansion $f = \sum a_j \phi_j$, where $\phi_j \in S_j$, and $\{\phi_j\}$ extends to an orthonormal basis of eigensections. If $B\phi_j = b_j \psi_j$ with b_j complex and $\|\psi_j\| = 1$, then $|b_j| \leqq \|B\|$, $\psi_j \in S_j$, and $\{\psi_j\}$ extends to an orthonormal basis of eigenfunctions. Since $Bf = \sum a_j b_j \psi_j$, we find Bf is analytic when f is.

Finally, if A is a positive semidefinite elliptic operator, then for each positive number L and each real number α there is a well defined positive operator $(A+L)^{\alpha}$ on $H^0(E)$. It is an easy consequence of the above theorem that, if A is analytic, then $(A+L)^{\alpha}$ maps analytic functions into analytic functions, and in fact the map is continuous and invertible in appropriate topologies. In particular, the operator $\Lambda = (L - \Delta)^{1/2}$ constructed in [6] has this property.

Alternate proof. We can also base the proof of our main theorem on a result of Kotake and Narasimhan [3]. This result uses the technique of [2], and applies to an elliptic operator on an arbitrary open set in Euclidean space. We show that our criterion for analyticity is equivalent to the criterion of [3] applied to a compact manifold. Letting again $A\phi_j=\lambda_j\phi_j$ and $\mu_j^m=|\lambda_j|$, the criterion of [3] for analyticity of $f=\sum f_j\phi_j$ is: there is a constant C such that for all $k\geqq 0$

$$\sum \mu_j^{2km}|f_j|^2 \leqq ((km)!)^2 C^{2k+2}. \tag{3}$$

The equivalence of (3) with our main theorem reduces easily to the following:

LEMMA. *Let* $0<\mu_j<\infty$, *and suppose* $\sum r^{\mu_j}<\infty$ *for each* $r<1$. *Then the condition* (3) *on sequences* $\{f_j\}$ *is equivalent to*

$$|f_j| \leqq Dt^{\mu_j} \quad \textit{for some } D<\infty \textit{ and } t<1. \tag{4}$$

Note that the condition $\sum r^{\mu_j}<\infty$ has been derived from (1) in the course of our original proof, when $(\mu_j)^m$ is the jth eigenvalue of A.

To prove the lemma, assume (3). Then each term of the series in the left of (3) is bounded by the right of (3), so $(\mu_j/k)^k|f_j|^{1/m}\leqq(mC^{1/m})^kC^{1/m}$. Stirling's formula gives, for an appropriate constant B, $|f_j|^{1/m}(\mu_j)^k/k!\leqq(B/2)^{k+1}$, so that $|f_j|^{1/m}\sum(\mu_j/B)^k/k!\leqq B$, i.e. $|f_j|\leqq B^m(e^{-1/B})^{\mu_j}$, which is (4).

For the converse, we assume (4) and prove $\sum\mu_j^k|f_j|\leqq k!C^{k+1}$, which implies (3). Consider $\psi(z)=\sum e^{z\mu_j}$. Since $\sum r^{\mu_j}<\infty$ for each $r<1$, ψ is analytic for $\mathrm{Re}(z)<0$, and thus on the compact set $\{z=\log t\}$ satisfies $|\psi^{(k)}(z)|\leqq k!C^{k+1}$, where $\psi^{(k)}$ is the kth derivative of ψ. Using this, we find $\sum\mu_j^k|f_j|\leqq D\sum\mu_j^k t^{\mu_j}=D\psi^{(k)}(\log t)\leqq k!C^{k+1}$.

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