ON DEDEKIND'S PROBLEM: THE NUMBER OF MONOTONE BOOLEAN FUNCTIONS

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The problem of determining the number $\psi(n)$ of elements of the free distributive lattice on n generators was posed by Dedekind [1] in 1897. It was solved by that author for $n=4$. R. Church [2] in 1940 and M. Ward [3] in 1946 obtained solutions for $n=5$ and $n=6$ respectively.

In 1954 E. N. Gilbert [4] showed that $\psi(n)$ satisfied the inequalities

$$2^{C_{n,[n/2]}} \leqq \psi(n) \leqq n^{C_{n,[n/2]+2}}$$

(where $C_{n,[n/2]}$ is the binomial coefficient).

Korobkov [5] in several papers published in 1962–1965 was able to improve the upper bound in $\psi(n)$ to

$$2^{4.23C_{n,[n/2]}}.$$

In 1966 G. Hansel [6] reduced the upper bound still further to

$$3^{C_{n,[n/2]}}.$$

In this paper we show that $\log_2\psi(n)$ is asymptotic to $C_{n,[n/2]}$; in fact we show that

$$2^{(1+\alpha_n)C_{n,[n/2]}} \leqq \psi(n) \leqq 2^{(1+\beta_n)C_{n,[n/2]}}$$

with $\alpha_n = ce^{-n/4}$, $\beta_n = c'(\log n)/n^{1/2}$.

The number $\psi(n)$ is equal to the number of ideals, or of antichains, or of monotone increasing functions into 0 and 1 definable on the lattice of subsets of an n-element set S_n. Here an ideal is a collection I of subsets such that $B\in I$, $A\subset B$ implies $A\in I$; an antichain is a collection of subsets no two of which are ordered by inclusion. An ideal can be uniquely corresponded to an antichain (its maximal members) and to a monotone function (that which takes the value 0 inside it and 1 outside it).

Hansel proceeds by partitioning the subsets of S_n into $C_{n,[n/2]}$ chains (totally ordered collections of subsets) in a certain manner. He then defines all possible monotone functions on each chain in turn.

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Since there turn out to be at most three possible ways to define functions in each chain, he obtains the result

$$\psi(n) \leqq 3^{C_{n,[n/2]}}.$$

We also partition the subsets into chains and define monotone functions on these in turn. We, however, only allow two possible definitions on most (asymptotically all) of the chains. In order to make up for the fact that such a procedure does not yield all possible functions, we repeat it using $n!$ different partitions into chains and a number α_n of different orderings of the chains. We then prove that every function is constructed at least once. Thus we obtain a limitation on $\psi(n)$ that is of the form

$$\psi(n) \leqq n!\alpha_n 2^{C_{n,[n/2]}\cdot(1+o(1))}.$$

Below a construction procedure for monotone (0, 1) functions on the subsets of an n-element set is presented. That every function is constructed by it at least once is then proven.

For convenience we first construct monotone functions on subsets having at least $n/2-\tau$ and at most $n/2+\tau$ elements in them. The same construction can be applied for other ranges of subset size; since every monotone function is monotone within each range, the total number of such functions cannot exceed the product of the number of such functions obtained for each range.

Consider some particular partition P of the subsets of size in the range described above into $C_{n,[n/2]}$ chains. (We can insist that each member of these chains except for the smallest cover another member.) Such partitions exist according to Dilworth's and Sperner's theorems and Hansel provides an example of one. Let π be any permutation of the n elements of S_n. π induces a permutation among subsets of S_n and hence among chains and among partitions into chains. The construction procedure described below will be applied to all $n!$ partitions obtained by applying each permutation π to the given partition P.

The construction procedure to be described involves ordered partitions into chains, that is to say partitions as in the last paragraph in which the blocks or chains are arranged in a specific order. For each partition $\pi(P)$ we will actually consider a number of different orderings of the chains. We first define a particular ordering as follows. Each chain contains one $[n/2]$ element set; we will define an ordering for these and order the chains accordingly. We order the $[n/2]$ element sets $A_1, A_2, \cdots, A_{C_{n,[n/2]}}$ in such a manner that as far as pos-

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sible, the set A_j satisfies $|A_j \cap A_k| < n/2 - \tau$ for all k in the range $j - l \leqq k < j$. Since there are at most

$$l \sum_{r=0}^{\tau} [C_{[n/2],r} C_{n-[n/2],r}]$$

subsets which A_j cannot be by this restriction it can always be arranged that A_j satisfies the given condition for

$$j \leqq j_0 \equiv C_{n,[n/2]} - l \sum_{r=0}^{\tau} C_{[n/2],r} C_{n-[n/2],r}.$$

Let the integers $j \leqq j_0$ be divided into blocks of size l; let the integers between j_0 and $C_{n,[n/2]}$ be considered individual blocks. We consider as the ordering of chains in the ordered partitions into chains used in the construction below, all

$$\left(\left[\frac{j_0}{l}\right] + 1 + l \sum_{r=0}^{\tau} C_{[n/2],r} C_{n-[n/2],r}\right)!$$

orderings of the blocks of chains associated with the blocks of A_j's just defined. For each n and τ we choose l so as to minimize the factorial just described; the minimum value of the factorial being denoted by $\alpha(n, \tau)$.

For each of the $n!\alpha(n, \tau)$ ordered partitions into chains apply the following procedure. On each chain in order assign the values 0 or 1 to each subset in any manner consistent with the definition previously chosen on earlier chains and with the monotone increasing property of the functions to be constructed. However consider only those functions for which 1's are assigned to the second largest chain members whose value is not predetermined by assignments on prior chains in at most t chains. The number of functions constructed according to this prescription cannot exceed

$$C_{n,[n/2],t}(2\tau)^t 2^{C_{n,[n/2]-t}} \tag{I}$$

since on all but the t chains in which this second largest unpredetermined member can be assigned 1, only two possible functions can be defined: the largest nonpredetermined set can be assigned 0 or 1.

The desired result (on the range $n/2 - \tau$ to $n/2 + \tau$) will be proven if we show that for some choice of t such that

$$n!\alpha(n, \tau) C_{n,[n/2],t}(2\tau)^t 2^{C_{n,[n/2]-t}} = 2^{C_{n,[n/2]}\,(1+o(1))}$$

all monotone functions will be constructed by the procedure described above applied to each ordered partition described above.

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Consider a monotone function f on subsets of S_n which have between $[n/2]-\tau$ and $[n/2]+\tau$ elements. Divide the subsets x satisfying $f(x)=1$ into two classes, with $x\in A_k$ if of the subsets y covered by x no more than k satisfy $f(y)=1$, and $x\in B_k$ otherwise. If $x\in A_k$, then in at most a proportion $k/|x|$ of the partitions described above will the subset y covered by x in the chain containing x satisfy $f(y)=1$, while in at least a proportion $1-k|x|$ of the partitions the member y of x's chain covered by x will have $f(y)=0$. The latter will take place therefore for at least

$$n!(1-k/|x|)|A_k|$$

chains in unordered partitions. We can conclude that

$$(1-k/|x|)|A_k| \leqq C_{n,[n/2]}$$

and that $x\in A_k$, x covers y, $f(y)=1$ will occur on the average over all partitions in no more than

$$C_{n,[n/2]}k/|x|(1-k/|x|)^{-1}$$

chains.

There must therefore exist a partition into chains in which this situation occurs in no more than

$$C_{n,[n/2]}\left(\frac{k}{[n/2]-\tau}\right)\left(1-\frac{k}{[n/2]-\tau}\right)^{-1}$$

chains. Choose such a partition P_f. Let x below be a smallest member of a chain in P_f satisfying $x\in B_k$. $f(x)$ will be predetermined if any of the at least k chains containing subsets y covered by x satisfying $f(y)=1$ appear in the partition ordering before the chain containing x. The construction of blocks above insures that no two sets covered by x can lie in the same block of chains permuted in forming the orderings of partitions considered above. Thus the set x will be in the first of the k chains in the ordering containing such y for at most $1/k$ of the orderings. Thus x will be predetermined for at least a proportion $(k-1)/k$ of the orderings of the partition P_f.

Since there are at most $C_{n,[n/2]}$ members of B_k that are smallest chain members with this property, the average over all orderings of the number of chains containing an x on B_k such that $f(x)$ was not predetermined is at most $(1/k)C_{n,[n/2]}$. There must therefore exist some ordering O_f in which $f(x)$ is predetermined in B_k in all but $(1/k)C_{n,[n/2]}$ chains.

If in the construction of f on a given chain, the second largest

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unpredetermined member is assigned a value 1, the largest unpredetermined member must either

(a) be in A_k and cover a set satisfying $f(y)=1$,

(b) be in B_k and be unpredetermined.

By the remarks above, if the ordering O_f of the partition P_f is chosen, these alternatives can occur in at most

$$C_{n,[n/2]}\left(\frac{1}{k}+\frac{k}{[n/2]-\tau}\left(1-\frac{k}{[n/2]-\tau}\right)^{-1}\right)$$

chains. We conclude, that for

$$t = C_{n,[n/2]}\left(\frac{1}{k}+\frac{k}{[n/2]-\tau}\left(1-\frac{k}{[n/2]-\tau}\right)^{-1}\right)$$

the procedure described above must indeed produce every monotone function on the given range. The optimal value for k here is roughly $[[n/2]-\tau]^{1/2}$ so that we may choose

$$t = C_{n,[n/2]}\frac{2}{([n/2]-\tau)^{1/2}}\left(1+O\left(\frac{2}{n}\right)^{1/2}\right).$$

It is easy to compute that the factor $n!$ and $\alpha(n,\tau)$ are utterly negligible compared to (I) for $\tau\sim n^{\beta}(C_{n,[n/2]})$, $\beta<1$ and that the number of monotone functions in the range considered therefore cannot exceed

$$2^{C_{n,[n/2]}(1+(c/n^{1/2})\log n)}$$

the second term in the exponent arising from evaluation of (I) with the value of t given above.

Similar arguments can be applied to compute the number of monotone functions on other ranges of subset size. By suitable choice of subset sizes, it can be shown that consideration of such subsets gives rise to a contribution to the total number of monotone functions that is very much smaller than is given by the second term above. We conclude that

$$\psi(n) \leqq 2^{C_{n,[n/2]}(1+(c/n^{1/2})\log n)}.$$

We can easily produce $2^{C_{n,[n/2]}}$ antichains by considering all collections of $[n/2]$ element subsets of S_n. Most such antichains will contain approximately half of all $[n/2]$ element subsets, and will contain subsets that are contained in all but approximately $(C_{n,[n/2]}+1)e^{-n/4}$ element sets. Any of the remaining $([n/2]+1)$ element sets can be added to the antichain. We can easily see therefore that

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$$\psi(n) \geqq 2^{C_{n,[n/2]}(1+e^{-n/4})}.$$

This lower bound can be improved [7].

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