SOME CRITERIA FOR NILPOTENCY IN GROUPS AND LIE ALGEBRAS

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We shall say that an automorphism α is *nilpotent* or *acts nilpotently* on a group G if in the holomorph $H=[G](\alpha)$ of G with α, α is a bounded left Engel element, that is, $[H, k\alpha]=1$ for some natural number k. Here $[H, k\alpha]$ means $[H, (k-1)\alpha]$ with $[H, 0\alpha]$ denoting H.

Let G' denote the commutator subgroup $[G, G]$, and let $\Phi(G)$ denote the Frattini subgroup of G. If α is an automorphism of a nilpotent group G such that the automorphism $\bar{\alpha}$ induced by α on G/G' is nolpotent (or with certain restrictions on the exponent of G on $G/\Phi(G)$), then by a well-known theorem of Philip Hall (cf. [**6**, p. 202]), α is nilpotent. Here we shall show that the same conclusion follows if we know that the restriction of α to a suitable subgroup of a nilpotent group is nilpotent. We prove the following two theorems announced in [**7**].

THEOREM 1. *Let G be a nilpotent group, let α be an automorphism of G, let F be a subgroup of G stable under α and such that α is nilpotent on F. If F contains its centralizer $C_G(F)$, then α is nilpotent on G.*

For the statement of Theorem 2 it will be convenient to say that a nilpotent group G is *of height k* if k is the least nonnegative integer so that for each prime p and each p-element g of G, $g^{p^k}\in G'$.

THEOREM 2. *Let G be a nilpotent group of height k, let α be an automorphism of G, let F be a subgroup of G stable under α and such that α is nilpotent on F. Suppose that F contains the elements of order 4 of $C_G(F)$, the elements of order p of $C_G(F)$ for all odd primes p, and the torsion-free elements of $C_G(F)$. Then α is nilpotent on G.*

Theorem 2 includes as a special case a recent result of Blackburn (cf. [**1**]).

In view of the known results about Engel elements we have the following consequence.

COROLLARY 1. *Let F be a nilpotent normal subgroup of a group G and suppose that G is either finite or else solvable with nilpotent Hirsch-Plotkin radical H, and suppose further that either $F\geqq C_G(F)$ or that F*

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is as in Theorem 2. If for each $x \in G$, the inner automorphism α_x determined by x induces a nilpotent automorphism on F, then G is nilpotent.

Since Theorem 1 is very closely related to a theorem of Thompson (cf. [**3**, p. 185]) we include the following generalization of the latter.

THEOREM 3. *A nilpotent group G of finite height has a characteristic subgroup C with the following properties:*

(i) *$C/Z(C)$ has height at most one provided G is periodic.*

(ii) *$[G, C]$ is contained in the center $Z(C)$ of C (and hence C has class at most two).*

(iii) $C_G(C) = Z(C)$.

(iv) *Every nonnilpotent automorphism of G induces a nonnilpotent automorphism of C.*

We also develop similar ideas for Lie algebras as follows. We say that a derivation δ is *nilpotent* or *acts nilpotently* on a Lie algebra L if in the holomorph $H = L + \{\delta\}$ of L with δ, δ is an Engel element. Then we have for Lie algebras the following analogues to our results for groups:

THEOREM 4. *Let F be a subalgebra of a nilpotent Lie algebra L such that F contains its centralizer $C_L(F)$. If δ is a derivation of L which maps F onto F and is nilpotent on F then δ is nilpotent on L.*

COROLLARY 2. *Let F be an ideal of a finite-dimensional Lie algebra L such that F contains $C_L(F)$. If for each x in L the inner derivation δ_x determined by x induces a nilpotent derivation on F, then L is nilpotent.*

THEOREM 5. *A nilpotent Lie algebra L has a characteristic subalgebra C with the following properties:*

(i) *$[L, C]$ is contained in the center $Z(C)$ of C and hence C has class at most two.*

(ii) $C_L(C) = Z(C)$.

(iii) *Every nonnilpotent derivation of L induces a nonnilpotent derivation of C.*

PROOFS. We note that easy induction arguments immediately give the following:

(1) The holomorph of a nilpotent group with a nilpotent automorphism is nilpotent.

(2) A nilpotent automorphism of a nilpotent group of finite exponent (of exponent p^k) has finite order (of order a power of p).

(3) If G has a normal series $G = G_0 > G_1 > \cdots > G_n = 1$ and an automorphism α so that for $i = 1, 2, \cdots, h$, α maps G_i onto G_i and

if α is nilpotent on each factor group G_{i-1}/G_i, then α is nilpotent on G.

PROOF OF THEOREM 1. We consider the case first where $G' \leqq F$. Since $[F, kG]=1$ and $[F, m\alpha]=1$ for appropriate natural numbers k and m, it follows that F has a normal series $F=F_0>F_1> \cdots >F_n=1$ so that if H denotes the holomorph $[G](\alpha)$ then $[F_i, H] \leqq F_{i+1}$ for $i=0, 1, \cdots, n-1$. Let C_i denote $C_G(F_i)$. Then $C_0 \leqq C_1 \leqq \cdots \leqq C_n = G$ and $C_0 \leqq F$. If C_{r+1} is the least of the C_i not in F we shall show that α is nilpotent on C_{r+1} as follows: $[F_r, C_{r+1}] \leqq F_r$ so that $[[F_r, C_{r+1}], m\alpha] = 1$. Then for $c \in C_{r+1}$, $f \in F_r$,

$$(*) \qquad [[f, c^{-1}], \alpha]^c[[c, \alpha^{-1}], f]^\alpha[[\alpha, f^{-1}], c]^f = 1$$

(cf. (*), p. 201 of [6]), and hence $[[f, c^{-1}], \alpha]=[f, [c, \alpha^{-1}]]^\alpha = [f^\alpha, [\alpha, c]]$. It follows that $[[[f, c^{-1}], \alpha], \alpha]=[[f^\alpha, [\alpha, c]], \alpha] = [f^{\alpha^2}, [\alpha, [c, \alpha]]]$ and $[[f, c^{-1}], j\alpha]=[f^{\alpha j}, [\alpha, [c, (j-1)\alpha]]]$ for each $j>1$. Since $[[F_r, C_{r+1}], m\alpha]=1$ it follows that $[C_{r+1}, m\alpha] \leqq C_r \leqq F$ and $[C_{r+1}, 2m\alpha]=1$. Hence α is nilpotent on C_{r+1} and consequently on FC_{r+1}. An induction then gives that α is nilpotent on $C_n=G$ and the statement of the theorem is proved in the case where $G' \leqq F$. Now let G^2 denote G' and for $t>2$, let G^t denote $[G^{t-1}, G]$ so that $G>G^2> \cdots >1$. In the general case suppose that G^t is the least member of the lower central series not in F. Then α is nilpotent on FG^t by what was shown above and an induction gives that α is nilpotent on G. This proves Theorem 1.

PROOF OF THEOREM 2. Let $G=G_0>G_1> \cdots >G_n=1$ be an invariant series of G which includes the members of the lower central series of G and so that each factor G_i/G_{i+1} has height at most 1. Let r be maximal so that $G_{r+1} \leqq F$ and assume inductively that the theorem is true for all $s<r$. Now $F \cap C_G(F)$ is central in $C_G(F)$ and the hypotheses of the theorem hold for $F \cap C_G(F)$ in $C_G(F)$. If we can show that α is nilpotent on $C_G(F)$, then by (3) α will be nilpotent on $FC_G(F)$, and by Theorem 1, α will be nilpotent on G.

Accordingly we need only consider the case where F is central in G. Then all the torsion free elements of G are in F and we let k be maximal so that F contains all the elements of order p^k of G for all p ($k>0$ by hypothesis). Let c be a p-element of G_r for some p; since α is nilpotent on F, a suitable pth power β of α is the identity on the Sylow p-subgroup F_p of F by (2). Then $[c, \beta]^{p^k}=(c^{-1}c^\beta)^{p^k}$ and since $[c^{-1}, c^\beta]$ is in G_r, hence in F and therefore central,

$$(c^{-1}c^\beta)^{p^k} = c^{-p^k}c^{\beta p^k}[c^{-1}, c^\beta]^{C_{p^k,2}}$$

(where $C_{p^k,2}$ is the binomial coefficient), which is $c^{-p^k}c^{\beta p^k}$ since

$$[c^{-1}, c^\beta]^{C_{p^k,2}} = [c^{-p}, c^\beta]^{C_{p^k,2}/p} = 1$$

(for $k=1$ it is only necessary to consider odd p). Thus $[c,\beta]^{p^k}=[c^{p^k},\beta]=1$, and hence $[c,\beta]\in F$. Since β is a pth power of α it follows that α (modulo the centralizer of c in (α)) and c generates a p-subgroup whose order is bounded in terms of the class of nilpotency and height of G independent of the element c. Thus α is nilpotent on $F_p\cap G_r$. Since this is true for each p, α is nilpotent on G_r and hence by (3) on FG_r. By the induction assumption α is nilpotent on G and the theorem is proved.

PROOF OF COROLLARY 1. Suppose first that G is finite. An induction argument on order gives that all maximal subgroups of G containing F are nilpotent. Hence G/F is solvable and thus G is solvable. We now consider the case where G is solvable with nilpotent Hirsch-Plotkin radical H and consider the subgroups $F\leqq H\leqq K\leqq G$ where K is a normal subgroup of G with K/H abelian. Then for $x\in K$, α_x is nilpotent on H by the theorems and hence x is a left Engel element of G. It follows from Theorem 4 of [4] that x is in H and therefore $K\leqq H$; since G is solvable it follows that $G\leqq H$ and hence that G is nilpotent, as was to be shown.

PROOF OF THEOREM 3. Theorem 1 includes the implication that condition (iii) of Theorem 3 implies condition (iv). Accordingly we need only prove that conditions (i), (ii), and (iii) hold for G. We let D be a maximal characteristic abelian subgroup of G and let its centralizer $C_G(D)$ be denoted by H. We let K be the complete inverse image of the maximal subgroup of height one of the center of G/D when G is periodic, while for G not periodic K will be the complete inverse image of the center of G/D (so that $[G,K]\leqq D$). We then let C be $H\cap K$ and Q be $C_G(C)$, noting that $Q\leqq H$ since $D\leqq C$. Since all the above subgroups are characteristic in G, $D(C\cap Q)$ is characteristic as well as abelian, so that from the maximality of D it follows that $C\cap Q\leqq D$ and therefore $H\cap K\cap Q=K\cap Q\leqq D$. Furthermore, since $C\leqq C_G(D)$, it follows that D is in C and from the maximality of D that D is in fact the center of C. Finally the fact that $K\cap Q\leqq D$ implies that $Q\leqq D$; for in the contrary case, modulo D, Q would be a nontrivial normal subgroup which did not meet the maximal subgroup of height one of the center of G (modulo D). Thus conditions (i), (ii), and (iii) are satisfied for C and the proof of the theorem is complete. It is worthy of notice that in case $C_G(D)\leqq D$, then D itself satisfies the conditions of the theorem in place of C.

PROOFS OF THEOREMS 4 AND 5. The proof of Theorem 4 is essentially the same as that of Theorem 1 except that the argument to replace the lines following (*) is less complicated, since the Jacobi identity is similar to but simpler than (*). For the proof of Theorem 5 we let D be a maximal characteristic subalgebra and let K be the complete inverse image of the center of L/D and then proceed as in Theorem 3. It should be remarked here that in a similar fashion the arguments on p. 202 of [6] for groups can be recast directly to give the analogous results for Lie algebras (cf. [2]).

PROOF OF COROLLARY 2. L induces a nilpotent algebra of linear transformations on the vector space F/F' and hence by Engel's theorem (cf. [5] for instance) L/F is nilpotent. Thus L is solvable. By an induction argument FL' is nilpotent and hence by Theorem 4 every $x \notin LF'$ is an Engel element of L. Then by Engel's theorem again, L is nilpotent.

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