ARCS IN INVERSE LIMITS ON [0, 1] WITH ONLY ONE BONDING MAP

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1. Introduction. In this note, by a continuum we mean a non-degenerate, compact, connected metric space. It is known ([2] or [5]) that a continuum is chainable if and only if it is homeomorphic to the inverse limit of a sequence of maps from [0, 1] onto [0, 1]. Indeed, the mappings may be required to be piecewise linear. Henderson has shown that a pseudo arc can be obtained as an inverse limit on [0, 1] with only one bonding map [3], and the author has shown that not every chainable continuum can be so obtained [4]. We now show that if \( M \) is an inverse limit on [0, 1] with only one bonding map, and the bonding map is piecewise monotone, then every subcontinuum of \( M \) contains an arc. Thus the set of all such continua is a proper subset of the set of chainable continua which are inverse limits on [0, 1] with only one bonding map.

2. Definitions and notation. If each term of the sequence \( g = \{g_i\} \) maps \([0, 1]\) onto \([0, 1]\), then the inverse limit of \( g \), denoted by \( \lim g \), is the subspace of the infinite cartesian product \([0, 1]^\omega\) consisting of all number sequences \( \{x_i\} \) such that for each \( i > 0 \), \( g_i(x_{i+1}) = x_i \). If \( f \) maps \([0, 1]\) onto \([0, 1]\), then \( \lim f \) denotes \( \lim g \) where \( g = f, f, \ldots \).

By an interval we mean a nondegenerate closed subinterval of \([0, 1]\), and the statement that \( A \) is an inverse sequence (for \( f \)) means that \( A \) is a sequence \( A_1, A_2, \ldots \) such that if \( i > 0 \), \( A_i \) is degenerate or an interval and \( f(A_{i+1}) = A_i \). The set of all points of \( \lim f \) such that for each \( i > 0 \), \( x_i \subseteq A_i \) is denoted by \( \lim(f, A) \). By a subinverse sequence of \( A \) is meant an inverse sequence \( B = \{B_i\} \) such that if \( i > 0 \), \( B_i \subseteq A_i \).

If each of \( I \) and \( I' \) is an interval, the statement that \( f \) maps \( I \) onto \( I' \) efficiently means that \( f(I) = I' \) and no interior point of \( I \) maps onto an endpoint of \( I' \).

3. Arcs in chainable continua. Our object in this section is to prove the following:

**Theorem.** If \( f \) is a piecewise monotone function from \([0, 1]\) onto \([0, 1]\) then each subcontinuum of \( \lim f \) contains an arc.

**Proof.** Suppose there is a subcontinuum of \( \lim f \) which contains no arc. Then there is an inverse sequence \( A = \{A_i\} \) such that \( \lim(f, A) \) contains no arc. There is an increasing number sequence \( x_0 = 0, x_1, \ldots, x_n, \ldots \)

Received by the editors June 3, 1968.
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\cdots, x_n = 1 \text{ such that if } 0 < i \leq n, f \bigl[ x_{i-1}, x_i \bigr] \text{ is monotone. Let } M \text{ denote the set of images under } f \text{ of the numbers in this sequence and let } \varepsilon \text{ denote a positive number such that if each of } I \text{ and } I' \text{ is an interval, and } f \text{ maps } I \text{ onto } I' \text{ efficiently, and the length of } I' \text{ is less than } \varepsilon, \text{ then } f \text{ is monotone on } I. \text{ Let } K \text{ denote the subset of } [0, 1] \text{ to which a number } x \text{ belongs if and only if there is a point } \{ p_i \} \text{ in lim } f \text{ such that } p_1 = x \text{ and such that either (1) for some } i > 0, p_i \in M \text{ or (2) for some } i > 0, f^{-1}(p_i) \text{ contains an interval. } K \text{ is countable and so there is a finite collection } G' \text{ of nonoverlapping intervals filling up } [0, 1] \text{ such that each interval in } G' \text{ is of length less than } \varepsilon/2, \text{ and no endpoint of an interval in } G', \text{ other than 0 or 1, is a number in } K. \text{ Let } G \text{ denote the intervals in } G' \text{ which have neither 0 nor 1 as an endpoint. We now state an easily established lemma.}

\text{Lemma 1. If } g \text{ maps } [0, 1] \text{ onto } [0, 1] \text{ and } \alpha = \{ \alpha_i \} \text{ is an inverse sequence for } g, \text{ and } I \text{ is a subinterval of a term } \alpha_i \text{ of } \alpha, \text{ then there is a subinverse sequence } \{ \beta_i \} \text{ of } \alpha \text{ such that } \beta_n = I \text{ and for each } i > n, g \text{ maps } \beta_{i+1} \text{ onto } \beta_i \text{ efficiently.}

\text{Applying Lemma 1, there is a subinverse sequence } A^0 = \{ A^0(i) \} \text{ of } A \text{ and an integer } n_0 \text{ such that for each } i \geq n_0, f \text{ maps } A^0(i+1) \text{ efficiently onto } A^0(i). \text{ If there is an } n \text{ such that for } i > n, f \text{ is monotone on } A^0(i), \text{ then } \lim(f, A^0) \text{ is an arc (see [1]). Since } \lim(f, A^0) \text{ is not an arc, there is a subsequence of } A^0 \text{ each term of which is of length at least } \varepsilon \text{ and so there is an interval } g_1 \text{ in } G \text{ which is a subset of each term of some subsequence of } A^0. \text{ Applying Lemma 1 again, there is a subinverse sequence } A^1 = \{ A^1(i) \} \text{ of } A^0 \text{ and an integer } n_1 \text{ such that } A^1(n_1) = g_1 \text{ and such that if } i \geq n_1, f \text{ maps } A^1(i+1) \text{ efficiently onto } A^1(i). \text{ As before we note that since } \lim(f, A^1) \text{ is not an arc, there is an interval } g_2 \text{ in } G \text{ which is a subset of each term of some subsequence of } A^1. \text{ We continue to establish the existence of a sequence } A^0, A^1, A^2, \cdots, \text{ an increasing integer sequence } n_1, n_2, \cdots \text{ and a sequence } g_1, g_2, \cdots \text{ of members of } G \text{ such that for each } i > 0, (1) A^i \text{ is a subinverse sequence of } A^{i-1}, (2) A^i(n_i) = g_i, (3) \text{ if } j \geq n_i, f \text{ maps } A^i(j+1) \text{ efficiently onto } A^i(j) \text{ and (4) } g_i \text{ is a subset of each term of a subsequence of } A^{i-1}. \text{ There is an integer } u \text{ and an integer } m < u \text{ such that } g_u = g_m. \text{ The sequence } A^m \text{ is a subinverse sequence of } A^0, A^m(n_m) = g_m, \text{ and } g_m \text{ is a subset of each term of a subsequence of } A^m. \text{ Further, as a consequence of the definition of } G \text{ we have that if } j \geq n_m, f \text{ is not constant on any interval which contains an endpoint of } A^m(j+1) \text{ and it follows that } A^m(j+1) \text{ is a component of } f^{-1}[A^m(j)]. \text{ Let } N \text{ denote an integer such that } A^m(n_m+N) \text{ contains } A^m(n_m). \text{ We shall show that if } j > 0, A^m(n_m+N+j) \text{ contains } A^m(n_m+j). \text{ Either } A^m(n_m+N+1)
\[\Delta A^m(n_m+1)\text{ or they do not intersect since } A^m(n_m+1)\text{ is a connected subset of } f^{-1}[A^m(n_m+N)]\text{ and } A^m(n_m+N+1)\text{ is a component of } f^{-1}[A^m(n_m+N)].\] Let \(k\) denote a positive integer such that \(A^m(n_m+N+k)\) contains \(g_m\). Since \(f^N(g_m) \subseteq g_m\), then \(f^N[A^m(n_m+N+k)]\) intersects \(g_m\) and thus intersects \(A^m(n_m+N+k)\). But

\[f^N[A^m(n_m+N+k)] = A^m(n_m+k),\]

so \(A^m(n_m+N+k)\) intersects \(A^m(n_m+k)\) which implies that \(A^m(n_m+N+1)\) intersects, and thus contains, \(A^m(n_m+1)\). It follows inductively that if \(j > 0\), \(A^m(n_m+N+j)\) contains \(A^m(n_m+j)\). Now let \(g = f^N\) and for each \(j > 0\) let \(B(j) = A^m(n_m+(j-1)N)\). Then \(g\) is a piecewise monotone map of \([0, 1]\) onto \([0, 1]\). \(\{B(i)\}\) is an inverse sequence (for \(g\)), and for each \(j > 0\), \(B(j+1) \supseteq B(j)\) and \(g\) maps \(B(j+1)\) efficiently onto \(B(j)\). Further, \(\lim(g, B)\) is homeomorphic to a subcontinuum of \(\lim(f, A)\) and so the following lemma applies to complete our proof.

**Lemma 2.** If \(g\) maps \([0, 1]\) onto \([0, 1]\) and is piecewise monotone and \(B = \{B_i\}\) is an inverse sequence (for \(g\)) such that for each \(i > 0\), \(B_i\) is a subinterval of \(B_{i+1}\) and \(g\) maps \(B_{i+1}\) efficiently onto \(B_i\), then \(\lim(g, B)\) contains an arc.

**Proof.** For each \(i > 0\), let \(B_i = [a_i, b_i]\). The sequence \(\{a_i\}\) is non-increasing and converges to a number \(a\). Similarly, \(\{b_i\}\) converges to a number \(b\). Further, there is a number \(x\) such that the point \((x, x, \cdots)\) is in \(\lim(g, B)\). Suppose that there is a positive integer \(n\) such that \(g(b_{n+1}) = a_{n+1}\) and \(g(a_{n+1}) = a_n\). According as \(g(b_{n+2}) = b_{n+2}\) or \(g(b_{n+2}) = a_{n+2}\), we have that either \(g([b_{n+2}, b_{n+3}])\) contains \(x\) and is nondegenerate or \(g([a_{n+2}, a_{n+3}])\) contains \(x\) and is nondegenerate. But \(g\) is piecewise monotone and there do not exist infinitely many mutually disjoint intervals such that if \(I\) is one of them, \(g(I)\) contains \(x\) and is nondegenerate. So there is an integer \(N'\) such that if \(n > N'\), then it is not true that \(g(b_{n+2}) = a_{n+1}\) and \(g(a_{n+1}) = a_n\). It follows that there is an integer \(N\) such that either (1) if \(n > N\), \(g(a_{n+1}) = a_n\), or (2) if \(n > N\), \(g(a_{n+1}) = b_n\). Suppose first that (1) holds and that \(b \neq x\). There is a number \(u, x < u < b\), such that \(g\) is monotone on \([u, b]\) and an integer \(k\) such that if \(j \geq k\), \(b_j > u\). If \(g(u) \leq u\), there is a point \(\{u_i\}\) in \(\lim g\) such that \(u_i = u\) and for \(i > 0\), \(u_i \leq u_{i+1} < b_{i+1}\). Then for \(i > 0\), \(g([u_{i+1}, b_{k+i}]) = [u_i, b_{k+i-1}]\) and \(g\) is monotonic on \([u_{i+1}, b_{k+i}]\). It follows that the set of all points \(\{p_i\}\) of \(\lim g\) such that for \(i > 0\), \(p_{k+i} \in [u_{i+1}, b_{k+i}]\) is an arc which is a subset of \(\lim(g, B)\). If \(g(u) > u\), then there is a number \(s\) and a number \(t\) such that (1) \(g(s) = s\), (2) \(x < s < t < u\), (3) if \(y \in [s, t]\), \(g(y) > y\) and (4) \(g\) is monotonic on \([s, t]\). There
is a point \( \{ t_i \} \) in \( \lim g \) such that \( t_1 = t \) and for each \( i > 0 \), \( s < t_{i+1} \leq t_i \). For each \( i > 0 \), \( g([s, t_{i+1}]) = [s, t_i] \) and \( g \) is monotonic on \([s, t_{i+1}]\), and it follows that the set of all points \( \{ p_i \} \) of \( \lim g \) such that for \( i \geq 0 \), \( p_{k+i} \in [s, t_{i+k}] \subseteq [a_{k+i}, b_{k+i}] \) is an arc which is a subset of \( \lim(g, B) \).

Thus we have that \( \lim(g, B) \) contains an arc in case there is an integer \( N \) such that if \( n > N \), \( g(a_{n+1}) = a_n \) and \( b \neq x \). An analogous argument suffices in case \( a \neq x \) and it remains to show that \( \lim(g, B) \) contains an arc in case there is an integer \( N \) such that if \( n > N \), \( g(a_{n+1}) = b_n \). In this case we consider the piecewise monotone map \( g^2 \) of \([0, 1]\) onto \([0, 1]\) and the inverse sequence \( B' = B_1, B_3, B_5, \ldots \). There is an integer \( k \) such that if \( i > k \), \( g^2(a_{2i+1}) = a_{2i-1} \), so an argument analogous to that given above applies to show that \( \lim(g^2, B') \) contains an arc.

But \( \lim(g, B) \) is homeomorphic to \( \lim(g^2, B') \) and this completes our proof.

References