INCLUSION THEOREMS FOR SONNENSCHEIN MATRICES[1]

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1. **Introduction.** Inclusion theorems for various methods of summability have been the subject of recent research [2], [5]. In this article necessary and sufficient conditions for the inclusion of Sonnenschein matrix methods are investigated with special attention to matrices with complex entries.

Let f be a function that is analytic for $z \in D_f = \{z: |z| < R\}$, $R > 1$ and $f(1) = 1$. Let

$$\{f(z)\}^n = \sum_{k=0}^{\infty} a_{nk} z^k \qquad n = 1, 2, \cdots,$$

$$a_{00} = 1, \quad a_{0k} = 0 \qquad k = 1, 2, \cdots.$$

Then f determines a sequence to sequence transformation, $A(f) = (a_{nk})$, whereby if $\{s_k\}$ is a sequence and $\sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k$, $n = 0, 1, 2, \cdots$ with $\sigma_n \to \sigma$ then $\{s_k\}$ is said to be $A(f)$-summable to σ. Such matrices are called Sonnenschein matrices [7]. Special well-known cases to be discussed here are the Taylor or Circle method, $T(r)$ [8], $f(z) = (1-r)z/(1-rz)$, $|r| < 1$; the Laurent method, $S(q)$ [8], $f(z) = (1-q)/(1-qz)$, $|q| < 1$; the Euler-Knopp method, $E(p)$ [1], $f(z) = (1-p) + pz$; and a generalization of the three preceding, the Karamata method, $K(\alpha, \beta)$ [7], $f(z) = \{\alpha + (1-\alpha-\beta)z\}/(1-\beta z)$, $|\beta| < 1$. In this new notation $T(r) = K(0, r)$, $S(q) = K(1-q, q)$, $E(p) = K(1-p, 0)$. Necessary and sufficient conditions for these methods to be regular are $0 \leqq r < 1$, [3]; $0 < q < 1$, [4]; $0 < p \leqq 1$, [1]; and $\alpha = \beta = 0$ or $1 - |\alpha|^2 > (1-\bar{\alpha})(1-\beta) > 0$, [6] respectively.

The following lemma and notation will be used in the sequel.

LEMMA 1. *Let $DA(f)$ denote the domain of values z for which the geometric series is $A(f)$-summable to $(1-z)^{-1}$. Then*

$$DA(f) = \{z: |f(z)| < 1\}, \qquad z \in D_f.$$

PROOF. Let the nth partial sum of the geometric series be denoted by $S_n = (1-z^{n+1})/(1-z)$. Then the $A(f)$-transform, $\{\sigma_n\}$, of $\{S_n\}$ is given by

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$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} S_k = (1-z)^{-1} \sum_{k=0}^{\infty} a_{nk} - (1-z)^{-1} \sum_{k=0}^{\infty} a_{nk} z^{k+1}$$
$$= (1-z)^{-1} - z(1-z)^{-1}\{f(z)\}^n.$$

Thus $\sigma_n \to (1-z)^{-1}$ if and only if $[f(z)]^n \to 0$ if and only if $|f(z)| < 1$. Let

$$m = \{x = \{x_n\} : x \text{ is bounded}\}, \qquad c = \{x = \{x_n\} : x \text{ is convergent}\},$$

$$c_{A(f)} = \left\{x : A(f)x = \left\{\sum_{k=0}^{\infty} a_{nk} x_k\right\} \in c\right\}, \qquad \bar{\Delta}(0, 1) = \{z : |z| \leqq 1\}.$$

2. Products and inverses.

THEOREM 1. *Suppose $A(f)$, $A(g)$ are Sonnenschein matrices and $g(\bar{\Delta}(0, 1)) \subset D_f$ then $A(f) \cdot A(g) = A(f \circ g)$ and moreover $(A(f)A(g))y = A(f)(A(g)y)$ for all $y \in m$.*

PROOF. Let $z \in \bar{\Delta}(0, 1)$ and $A(f) = (a_{nk})$, $A(g) = (b_{kj})$. Then

$$\{g(z)\}^k = \sum_{j=0}^{\infty} b_{kj} z^j, \qquad k = 0, 1, 2, \cdots$$

and the convergence is absolute. Since $g(z) \in D_f$

$$\{f(g(z))\}^n = \sum_{k=0}^{\infty} a_{nk}\{g(z)\}^k, \qquad n = 0, 1, 2, \cdots$$
(1)
$$= \sum_{k=0}^{\infty} a_{nk}\left(\sum_{j=0}^{\infty} b_{kj} z^j\right) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{nk} b_{kj}\right) z^j.$$

The rearrangement (1) is permitted since the series involved converge absolutely. Likewise $f(g(z))$ is analytic on $D_{f \circ g} \supset \bar{\Delta}(0, 1)$ and hence

$$\{f(g(z))\}^n = \sum_{k=0}^{\infty} c_{nk} z^k, \qquad n = 0, 1, 2, \cdots.$$

Thus by (1) and the uniqueness of power series representation

$$c_{nk} = \sum_{j=0}^{\infty} a_{nj} b_{jk}, \qquad n = 0, 1, 2, \cdots, \qquad k = 0, 1, 2, \cdots.$$

If $y \in m$, there exists M, such that $|y_j| < M$, for all j and

$$\sum_{k=0}^{\infty} a_{nk}\left(\sum_{j=0}^{\infty} b_{kj} y_j\right) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{nk} b_{kj}\right) y_j,$$

since

$$\sum_{k=0}^{\infty} | a_{nk} | \left(\sum_{j=0}^{\infty} | b_{kj} | \, | y_j | \right) \leqq M \sum_{k=0}^{\infty} | a_{nk} | \sum_{j=0}^{\infty} | b_{nj} |$$

and the right-hand side converges since $1 \in D_{f \circ g}$.

In the following corollary f is a one-to-one analytic function and $D_{f^{-1}}$ is the disk about the origin on which f^{-1} has a power series representation. (These two conditions are summarized in the single hypothesis $A(f)$ and $A(f^{-1})$ are Sonnenschein matrices.)

COROLLARY. *Suppose $A(f)$ and $A(f^{-1})$ are Sonnenschein matrices with $D_{f^{-1}} \supset f(\bar{\Delta}(0,1))$ then $A(f) \cdot A(f^{-1}) = I = A(f^{-1}) \cdot A(f)$, where I is the identity matrix, i.e. $A(f^{-1}) = \{A(f)\}^{-1}$.*

PROOF. $A(f)$ and $A(f^{-1})$ Sonnenschein imply $D_f \supset \bar{\Delta}(0,1)$ and $D_{f^{-1}} \supset \bar{\Delta}(0, 1)$. Furthermore, $D_f \supset f^{-1}(\bar{\Delta}(0, 1))$ and $D_{f^{-1}} \supset f(\bar{\Delta}(0,1))$. Therefore $A(f) \cdot A(f^{-1}) = A(f \circ f^{-1}) = A(e) = I = A(e) = A(f^{-1} \circ f) = A(f^{-1}) \cdot A(f)$, where e is the identity function on the domain of $f \circ f^{-1}$ and $f^{-1} \circ f$ respectively.

3. **Inclusion theorems.** The following theorem can easily be proved using infinite matrix algebra.

THEOREM 2. *Let A, B be one-to-one sequence matrix transformations. Let A^{-1} exist and $B(A^{-1}y) = (BA^{-1})y$ and $A(A^{-1}y) = (AA^{-1})y = y$, for all $y \in m$. Then $c_A \subset c_B$ if and only if BA^{-1} is conservative. Moreover, if A is regular then Ax and Bx converge to the same limit for $x \in c_A$ if and only if BA^{-1} is regular.*

We are now prepared to prove our main result.

THEOREM 3. *Suppose $A(f)$, $A(g)$ and $A(f^{-1})$ are Sonnenschein matrices with $A(f)$ a regular, one-to-one transformation. Then $c_{A(f)} \subset c_{A(g)}$ and $A(f)x$ and $A(g)x$ converge to the same limit if and only if $D_g \supset f^{-1}(\bar{\Delta}(0, 1))$ and $A(g) \cdot A(f^{-1})$ is regular.*

PROOF. *Sufficiency.* Since $A(f)$ is a regular Sonnenschein matrix and $A(f^{-1})$ is Sonnenschein, a result of Bajšanski [2] implies $f(\bar{\Delta}(0, 1)) \subset \bar{\Delta}(0,1) \subset D_{f^{-1}}$. Hence by the corollary to Theorem 1 $A(f^{-1}) = \{A(f)\}^{-1}$ and $A(g) \cdot A(f^{-1}) = A(g \circ f^{-1}) = A(g) \cdot \{A(f)\}^{-1}$ and $A(g)(A(f^{-1})y) = (A(g) \cdot A(f^{-1}))y$, $A(f)(A(f^{-1})y) = (A(f) \cdot A(f^{-1}))y = y$, for all $y \in m$. Thus Theorem 2 implies the result.

Necessity. By Theorem 2 it remains only to show that $D_g \supset f^{-1}(\bar{\Delta}(0, 1))$ is necessary. Suppose $c_{A(f)} \subset c_{A(g)}$ and $D_g \not\supset f^{-1}(\bar{\Delta}(0, 1)) = \{z : |f(z)| < 1, z \in D_f\}$. By Lemma 1 this implies $DA(g) \not\supset DA(f)$ and this contradicts the hypothesis $c_{A(f)} \subset c_{A(g)}$.

Let $f(z)=\{\alpha'+(1-\alpha'-\beta')z\}/(1-\beta'z)$ and

$$g(z) = \{\alpha + (1 - \alpha - \beta)z\}/(1 - \beta z)$$

with $|\beta|<1$ and $|\beta'|<1$. Then $A(f)=K(\alpha', \beta')$ and $A(g)=K(\alpha, \beta)$. When no confusion can arise $K(\alpha', \beta')\cap m\subset K(\alpha, \beta)$ will replace the more cumbersome $c_{K(\alpha',\beta')}\cap m\subset c_{K(\alpha,\beta)}$.

THEOREM 4. *Suppose* $|\beta|<1$, $|\beta'|<1$ *and*

$$\text{(i)}\qquad \begin{aligned} &\big|\, |\alpha'\beta' - 1 + \alpha' + \beta'| - 2|\alpha'\beta'|\cos\theta \,\big| \\ &\qquad \geqq 2|\beta'| \left| \frac{\alpha' + \beta' - 1 - \alpha'\beta'}{\alpha' + \beta' - 1 + \alpha'\beta'} \right| \end{aligned}$$

where θ *is the positive angle between* α' *and* $\mu=(\alpha'\beta'-1+\alpha'+\beta')/2\beta'$ *and*

$$\text{(ii)}\qquad 1 - |\alpha'|^2 > (1 - \bar{\alpha}')(1 - \beta') > 0 \quad \textit{or} \quad \alpha' = \beta' = 0$$

then $K(\alpha', \beta')\cap m\subset K(\alpha, \beta)$ *and the transformed limits are the same if and only if*

$$\text{(iii)}\qquad |\beta|\,\big|1 - |\alpha'|^2\big| < \big|\, |\beta'|^2 - |1 - \beta' - \alpha'|^2 \big|$$

and

$$\text{(iv)}\qquad \begin{aligned} &|(1 - \beta') - \alpha'(1 - \beta)|^2 - |\alpha(1 - \beta') - \alpha'(1 - \beta)|^2 \\ &\qquad > (1 - \bar{\alpha})(1 - \beta)(1 - \bar{\beta}')(1 - \alpha') > 0 \end{aligned}$$

or $\alpha=\alpha'$ *and* $\beta=\beta'$.

PROOF. If $|\beta|<1$, $|\beta'|<1$, then $A(f)=K(\alpha', \beta')$ and $A(g)=K(\alpha, \beta)$ are Sonnenschein matrices and moreover it follows that the $K(\alpha', \beta')$ transform is one-to-one on $c_{K(\alpha',\beta')}\cap m$. For $A(f^{-1})$ to be Sonnenschein it is necessary that the range of f include the unit disk, i.e.

$$f(\{z\colon |z| < 1/|\beta'|\}) \supset \bar{\Delta}(0, 1).$$

f transforms the disk $D(0, 1/|\beta'|)$ conformally onto the half plane, H, whose boundary contains $f(-1/\beta')=\mu$ and whose interior contains $f(0)=\alpha'$. The line through μ and α' is thus perpendicular to the boundary of H, because the line through $-1/\beta'$ and 0 is perpendicular to the circle $C(0, 1/|\beta'|)$. A simple calculation shows $H\supset\bar{\Delta}(0, 1)$ if and only if

$$|\mu|\,\big|\,|\mu| - |\alpha'|\cos\theta\big| \geqq |\mu - \alpha'|,$$

where $\mu=\{\alpha'\beta'-1+\alpha'+\beta'\}/2\beta'$, if and only if (i) holds.

$$f^{-1}(z) = (z-\alpha')/\{\beta' z + (1-\alpha'-\beta')\}$$

$$= \frac{\left(\dfrac{-\alpha'}{1-\alpha'-\beta'}\right) + \left(1 - \left[\dfrac{-\alpha'}{1-\alpha'-\beta'}\right] - \left[\dfrac{-\beta'}{1-\alpha'-\beta'}\right]\right) z}{1 - \left(\dfrac{-\beta'}{1-\alpha'-\beta'}\right) z};$$

thus by the corollary to Theorem 1,

$$\{A(f)\}^{-1} = A(f^{-1}) = K\left(\frac{-\alpha'}{1-\alpha'-\beta'}, \frac{-\beta'}{1-\alpha'-\beta'}\right) = K^{-1}(\alpha', \beta')$$

if $|\beta'| < 1$ and (i) holds.

By Theorem 3, if $D_g \supset f^{-1}(\bar{\Delta}(0, 1))$ then $A(g)\cdot A(f^{-1}) = A(g\circ f^{-1}) = K(\alpha, \beta)\cdot K^{-1}(\alpha', \beta')$ which implies

$$K(\alpha, \beta) K^{-1}(\alpha', \beta') = K\left(\frac{\alpha(1-\beta') - \alpha'(1-\beta)}{1-\alpha'-\beta'+\beta\alpha'}, \frac{\beta-\beta'}{1-\alpha'-\beta'+\beta\alpha'}\right).$$

Sledd [6] proved $K(\alpha^*, \beta^*)$ is regular if and only if $\alpha^* = \beta^* = 0$ or $1 - |\alpha^*|^2 > (1-\bar{\alpha}^*)(1-\beta^*) > 0$. Thus $K(\alpha', \beta')$ is regular if and only if (ii) holds and $K(\alpha, \beta)\cdot K^{-1}(\alpha', \beta')$ is regular if and only if (iv) holds.

Finally, $D_g \supset f^{-1}(\bar{\Delta}(0, 1))$ if and only if

$$(1) \quad \frac{\left|1 - |\alpha'|^2 - (1-\bar{\alpha}')(1-\beta')\right| + \left|(1-\bar{\alpha}')(1-\beta')\right|}{\left| |\beta'|^2 - |1-\alpha'-\beta'|^2 \right|} < \frac{1}{|\beta|},$$

since $f^{-1}(\bar{\Delta}(0, 1))$ is a disk with center,

$$C = \frac{1 - |\alpha'|^2 - (1-\alpha')(1-\bar{\beta}')}{|\beta'|^2 - |1-\alpha'-\beta'|^2}$$

and radius

$$R = \frac{|1-\beta'|\,|1-\alpha'|}{\left| |\beta'|^2 - |1-\alpha'-\beta'|^2 \right|}$$

and (1) is equivalent to $|C| + |R| < 1/|\beta|$. Thus the transformed disk is contained in $D_g = \{z: |z| < 1/|\beta|\}$ if and only if (1) holds. Thus if (ii) holds (iii) is equivalent to (1).

The following corollaries to Theorem 4 give necessary and sufficient conditions for inclusion of some well-known matrix transforms by other matrix methods. In particular they answer some questions posed by Schoonmaker [5].

COROLLARY 1. *If* $\frac{1}{2}<p\leqq 1$ *and* $0<r<1$, *then* $E(p)\subset T(r)$ *if and only if* $(1-p)/(2-p)<r<p/(2-p)$.

PROOF. With the notation of Theorem 4, $E(p)=K(1-p, 0)$, $T(r)=K(0, r)$. Thus conditions (i) and (ii) of that theorem are satisfied, because (i) is trivially true and (ii) is equivalent to $0<p\leqq 1$. Condition (iii) with hypothesis $\frac{1}{2}<p\leqq 1$ and $0<r<1$ becomes $|r|\,|1-|1-p|^2|<|p|^2$. This holds if and only if

$$r < p/(2-p). \tag{1}$$

Condition (iv) with $\frac{1}{2}<p\leqq 1$ and $0<r<1$ becomes

$$|1-(1-p)(1-r)|^2-|1(1-p)(1-r)|^2>(1-r)p>0$$

which is equivalent to

$$r > (1-p)/(2-p). \tag{2}$$

But inequalities (1) and (2) can hold simultaneously only if $\frac{1}{2}<p$ and thus the result follows.

COROLLARY 2. *If* $0\leqq r\leqq\frac{1}{3}$ *and* $|q|<1$ *then* $T(r)\cap m\subset S(q)$ *if and only if* $0<q<1-2r$.

PROOF. $S(q)=K(1-q, q)$ and $T(r)=K(0, r)$. If $0\leqq r\leqq\frac{1}{3}$ and $|q|<1$, condition (i) of Theorem 4 is satisfied, and (ii) is satisfied since (ii) is equivalent to regularity of $T(r)$ or $0\leqq r<1$. Conditions (iii) and (iv) will be satisfied if and only if

$$|q| < |\,|1-r|^2-|r|^2| \tag{1}$$

and

$$|1-r|^2-|(1-q)(1-r)|^2>\bar{q}(1-q)(1-r)>0. \tag{2}$$

It follows from $\bar{q}(1-q)(1-r)>0$ and $|q|<1$ that q is real and $q>0$. But then, under the hypothesis of the corollary, (1) becomes $q<1-2r$ and (2) becomes $q>(2r-1)/r$. But this latter inequality is satisfied since $(2r-1)/r<0<q$. Therefore (1) and (2) are equivalent to $0<q<1-2r$.

COROLLARY 3. *If* $0\leqq r\leqq\frac{1}{3}$, *then* $T(r)\cap m\subset E(p)$ *if and only if* $0<p<1$.

PROOF. $K(0, r)=T(r)$ and $K(1-p, 0)=E(p)$. If $0\leqq r\leqq\frac{1}{3}$ then, a fortiori, $0\leqq r<1$ which is equivalent to condition (ii) and implies condition (i) of Theorem 4. Conditions (iii) and (iv) of that theorem are equivalent to

$$(1) \qquad 0 < \big|\, |r|^2 - |1-r|^2 \,\big|,$$

and

$$(2) \qquad |1-r|^2 - |(1-p)(1-r)|^2 > \bar{p}(1-r) > 0,$$

respectively. (1) is trivially satisfied and (2) implies p is real, $p>0$. The first inequality in (2) thus reduces to $p<1$ and the result follows.

Corollaries 1, 2, and 3 strengthen and add new results to theorems of Schoonmaker [**5**]. In conclusion it should be noted that results for $E(p)\supset S(q)$ and $T(r)\supset S(q)$ could not be found using the methods of this paper because $f^{-1}(z)=\{z-(1-q)\}/qz$, $f(z)=(1-q)/(1-qz)$, $|q|<1$ is not analytic at the origin. This leads the author to suspect that $S^{-1}(q)$ does not exist but no results along these lines could be found.

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