INCLUSION THEOREMS FOR SONNENSCHEIN MATRICES

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1. Introduction. Inclusion theorems for various methods of summability have been the subject of recent research [2], [5]. In this article necessary and sufficient conditions for the inclusion of Sonnenschein matrix methods are investigated with special attention to matrices with complex entries.

Let $f$ be a function that is analytic for $z \in D_f = \{ z : |z| < R \}, R > 1$ and $f(1) = 1$. Let

$$
\{ f(z) \}^n = \sum_{k=0}^{\infty} a_{nk} z^k, \quad n = 1, 2, \cdots,
$$

$$
a_{00} = 1, \quad a_{0k} = 0, \quad k = 1, 2, \cdots.
$$

Then $f$ determines a sequence to sequence transformation, $A(f) = (a_{nk})$, whereby if $\{ s_k \}$ is a sequence and $\sigma_n = \sum_{k=0}^{n} a_{nk} s_k, n = 0, 1, 2, \cdots$ with $\sigma_n \to \sigma$ then $\{ s_k \}$ is said to be $A(f)$-summable to $\sigma$. Such matrices are called Sonnenschein matrices [7]. Special well-known cases to be discussed here are the Taylor or Circle method, $T(r)$ [8], $f(z) = (1 - r) z / (1 - rz), |r| < 1$; the Laurent method, $S(q)$ [8], $f(z) = (1 - q) / (1 - qz), |q| < 1$; the Euler-Knopp method, $E(p)$ [1], $f(z) = (1 - p + pz);$ and a generalization of the three preceding, the Karamata method, $K(\alpha, \beta)$ [7], $f(z) = \{ \alpha + (1 - \alpha - \beta)z \} / (1 - \beta z), |\beta| < 1$. In this new notation $T(r) = K(0, r), S(q) = K(1 - q, q), E(p) = K(1 - p, 0).$ Necessary and sufficient conditions for these methods to be regular are $0 \leq r < 1, [3]; 0 < q < 1, [4]; 0 < p \leq 1, [1];$ and $\alpha = \beta = 0$ or $1 - |\alpha|^2 > (1 - \alpha)(1 - \beta) > 0, [6]$ respectively.

The following lemma and notation will be used in the sequel.

**Lemma 1.** Let $DA(f)$ denote the domain of values $z$ for which the geometric series is $A(f)$-summable to $(1 - z)^{-1}$. Then

$$
DA(f) = \{ z : |f(z)| < 1 \}, \quad z \in D_f.
$$

**Proof.** Let the $n$th partial sum of the geometric series be denoted by $S_n = (1 - z^{n+1}) / (1 - z)$. Then the $A(f)$-transform, $\{ \sigma_n \}$, of $\{ S_n \}$ is given by

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1 This work was part of the author's doctoral dissertation completed at Lehigh University, 1968, under the direction of Professor J. P. King.
\[ \sigma_n = \sum_{k=0}^{\infty} a_{nk} S_k = (1 - z)^{-1} \sum_{k=0}^{\infty} a_{nk} - (1 - z)^{-1} \sum_{k=0}^{\infty} a_{nk} z^{k+1} \]

\[ = (1 - z)^{-1} - z(1 - z)^{-1} \{ f(z) \}^n. \]

Thus \( \sigma_n \to (1 - z)^{-1} \) if and only if \( \{ f(z) \}^n \to 0 \) if \( n \to \infty \) and only if \( |f(z)| < 1 \).

Let

\[ m = \{ x = \{ x_n \} : x \text{ is bounded} \}, \quad c = \{ x = \{ x_n \} : x \text{ is convergent} \}, \]

\[ c_{A(f)} = \{ x : A(f)x = \left\{ \sum_{k=0}^{\infty} a_{nk} x_k \right\} \in c \}, \quad \Delta(0, 1) = \{ z : |z| \leq 1 \}. \]

2. Products and inverses.

**Theorem 1.** Suppose \( A(f), A(g) \) are Sonnenschein matrices and \( g(\Delta(0, 1)) \subseteq D_f \) then \( A(f) \cdot A(g) = A(f \circ g) \) and moreover \( (A(f)A(g))y = A(f)(A(g)y) \) for all \( y \in m \).

**Proof.** Let \( z \in \Delta(0, 1) \) and \( A(f) = (a_{nk}), A(g) = (b_{kj}) \). Then

\[ \{ g(z) \}^k = \sum_{j=0}^{\infty} b_{kj} z^j, \quad k = 0, 1, 2, \ldots \]

and the convergence is absolute. Since \( g(z) \in D_f \)

\[ \{ f(g(z)) \}^n = \sum_{k=0}^{\infty} a_{nk} \{ g(z) \}^k, \quad n = 0, 1, 2, \ldots \]

(1)

\[ = \sum_{k=0}^{\infty} a_{nk} \left( \sum_{j=0}^{\infty} b_{kj} z^j \right) = \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{nk} b_{kj} \right) z^j. \]

The rearrangement (1) is permitted since the series involved converge absolutely. Likewise \( f(g(z)) \) is analytic on \( D_{\text{range}} \supseteq \Delta(0, 1) \) and hence

\[ \{ f(g(z)) \}^n = \sum_{k=0}^{\infty} c_{nk} z^k, \quad n = 0, 1, 2, \ldots \]

Thus by (1) and the uniqueness of power series representation

\[ c_{nk} = \sum_{j=0}^{\infty} a_{nj} b_{kj}, \quad n = 0, 1, 2, \ldots, \quad k = 0, 1, 2, \ldots. \]

If \( y \in m \), there exists \( M \), such that \( |y_j| < M \), for all \( j \) and

\[ \sum_{k=0}^{\infty} a_{nk} \left( \sum_{j=0}^{\infty} b_{kj} y_j \right) = \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{nk} b_{kj} \right) y_j, \]

since
and the right-hand side converges since $1 \in D_{f^{-1}}$.

In the following corollary, $f$ is a one-to-one analytic function and $D_{f^{-1}}$ is the disk about the origin on which $f^{-1}$ has a power series representation. (These two conditions are summarized in the single hypothesis $A(f)$ and $A(f^{-1})$ are Sonnenschein matrices.)

**Corollary.** Suppose $A(f)$ and $A(f^{-1})$ are Sonnenschein matrices with $D_{f^{-1}} \supseteq f(\Delta(0, 1))$ then $A(f) \cdot A(f^{-1}) = I = A(f^{-1}) \cdot A(f)$, where $I$ is the identity matrix, i.e., $A(f^{-1}) = \{A(f)\}^{-1}$.

**Proof.** $A(f)$ and $A(f^{-1})$ Sonnenschein imply $D_{f} \supseteq \Delta(0, 1)$ and $D_{f} \supseteq \Delta(0, 1)$. Furthermore, $D_{f} \supseteq f^{-1}(\Delta(0, 1))$ and $D_{f} \supseteq f(\Delta(0, 1))$. Therefore $A(f) \cdot A(f^{-1}) = A(f \circ f^{-1}) = A(e) = I = A(e) = A(f^{-1} \circ f) = A(f^{-1}) \cdot A(f)$, where $e$ is the identity function on the domain of $f \circ f^{-1}$ and $f^{-1} \circ f$ respectively.

3. **Inclusion theorems.** The following theorem can easily be proved using infinite matrix algebra.

**Theorem 2.** Let $A, B$ be one-to-one sequence matrix transformations. Let $A^{-1}$ exist and $B(A^{-1}y) = (BA^{-1})y$ and $A(A^{-1}y) = (AA^{-1})y = y$, for all $y \in m$. Then $c_{A} \subseteq c_{B}$ if and only if $BA^{-1}$ is conservative. Moreover, if $A$ is regular then $Ax$ and $Bx$ converge to the same limit for $x \in c_{A}$ if and only if $BA^{-1}$ is regular.

We are now prepared to prove our main result.

**Theorem 3.** Suppose $A(f)$, $A(g)$ and $A(f^{-1})$ are Sonnenschein matrices with $A(f)$ a regular, one-to-one transformation. Then $c_{A(f)} \subseteq c_{A(g)}$ and $A(f)x$ and $A(g)x$ converge to the same limit if and only if $D_{f} \supseteq f^{-1}(\Delta(0, 1))$ and $A(g) \cdot A(f^{-1})$ is regular.

**Proof.** **Sufficiency.** Since $A(f)$ is a regular Sonnenschein matrix and $A(f^{-1})$ is Sonnenschein, a result of Bajšanski [2] implies $f(\Delta(0, 1)) \subseteq \Delta(0, 1) \subseteq D_{f^{-1}}$. Hence by the corollary to Theorem 1 $A(f^{-1}) = \{A(f)\}^{-1}$ and $A(g) \cdot A(f^{-1}) = A(g \circ f^{-1}) = A(g) \cdot \{A(f)\}^{-1}$ and $A(g)(A(f^{-1})y) = (A(g) \cdot A(f^{-1}))y$, $A(f)(A(f^{-1})y) = (A(f) \cdot A(f^{-1}))y = y$, for all $y \in m$. Thus Theorem 2 implies the result.

**Necessity.** By Theorem 2 it remains only to show that $D_{f} \supseteq f^{-1}(\Delta(0, 1))$ is necessary. Suppose $c_{A(f)} \subseteq c_{A(g)}$ and $D_{f} \supseteq f^{-1}(\Delta(0, 1)) = \{z : |f(z)| < 1, z \in D_{f}\}$. By Lemma 1 this implies $DA(g) \supseteq DA(f)$ and this contradicts the hypothesis $c_{A(f)} \subseteq c_{A(g)}$. 

Let \( f(z) = \frac{\alpha' + (1 - \alpha' - \beta')z}{(1 - \beta'z)} \) and
\[
g(z) = \frac{\alpha + (1 - \alpha - \beta)z}{(1 - \beta z)}
\]
with \(|\beta| < 1\) and \(|\beta'| < 1\). Then \( A(f) = K(\alpha', \beta') \) and \( A(g) = K(\alpha, \beta) \).
When no confusion can arise \( K(\alpha', \beta') \cap m \subseteq K(\alpha, \beta) \) will replace the more cumbersome \( c_{K(\alpha', \beta')} \cap m \subseteq c_{K(\alpha, \beta)} \).

**Theorem 4.** Suppose \(|\beta| < 1\), \(|\beta'| < 1\) and

\[
|\alpha'\beta' - 1 + \alpha' + \beta' - 2| \cdot |\alpha'\beta'| \cos \theta | \\
\geq 2 |\beta'| \cdot \left| \frac{\alpha' + \beta' - 1 - \alpha'\beta'}{\alpha' + \beta' - 1 + \alpha'\beta'} \right|
\]

where \( \theta \) is the positive angle between \( \alpha' \) and \( \mu = (\alpha'\beta' - 1 + \alpha' + \beta')/2\beta' \) and

\[
1 - |\alpha'|^2 > (1 - \alpha')(1 - \beta') > 0 \quad \text{or} \quad \alpha' = \beta' = 0
\]

then \( K(\alpha', \beta') \cap m \subseteq K(\alpha, \beta) \) and the transformed limits are the same if and only if

\[
|\beta| \cdot 1 - |\alpha'|^2 < |\beta'|^2 - |1 - \beta' - \alpha'|^2
\]

and

\[
(1 - \beta') - \alpha'(1 - \beta) |^2 - |(1 - \beta')(1 - \alpha')| (1 - \alpha') > 0
\]

or \( \alpha = \alpha' \) and \( \beta = \beta' \).

**Proof.** If \(|\beta| < 1\), \(|\beta'| < 1\), then \( A(f) = K(\alpha', \beta') \) and \( A(g) = K(\alpha, \beta) \) are Sonnenschein matrices and moreover it follows that the \( K(\alpha', \beta') \) transform is one-to-one on \( c_{K(\alpha', \beta')} \cap m \). For \( A(f^{-1}) \) to be Sonnenschein it is necessary that the range of \( f \) include the unit disk, i.e.

\[
f(\{z: |z| < 1/|\beta'| \}) \supset \Delta(0, 1).
\]

\( f \) transforms the disk \( D(0, 1/|\beta'|) \) conformally onto the half plane, \( H \), whose boundary contains \( f(-1/\beta') = \mu \) and whose interior contains \( f(0) = \alpha' \). The line through \( \mu \) and \( \alpha' \) is thus perpendicular to the boundary of \( H \), because the line through \( -1/\beta' \) and \( 0 \) is perpendicular to the circle \( C(0, 1/|\beta'|) \). A simple calculation shows \( H \supset \Delta(0, 1) \) if and only if

\[
|\mu| | | \mu | - |\alpha'| \cos \theta | \geq |\mu - \alpha'|,
\]

where \( \mu = \{\alpha'\beta' - 1 + \alpha' + \beta'/2\beta' \) if and only if (i) holds.
$f^{-1}(z) = (z - \alpha') / \{\beta'z + (1 - \alpha' - \beta')\}$

$$= \frac{-\alpha'}{1 - \alpha' - \beta'} + \left(1 - \left[\frac{-\alpha'}{1 - \alpha' - \beta'}\right] - \left[\frac{-\beta'}{1 - \alpha' - \beta'}\right]\right)z$$

$$\frac{1 - \left(\frac{-\beta'}{1 - \alpha' - \beta'}\right)}{z}$$

thus by the corollary to Theorem 1,

$$\{A(f)^{-1} - A(f^{-1}) = K\left(\frac{-\alpha'}{1 - \alpha' - \beta'}, \frac{-\beta'}{1 - \alpha' - \beta'}\right) = K^{-1}(\alpha', \beta')$$

if $|\beta'| < 1$ and (i) holds.

By Theorem 3, if $D_\delta \supseteq f^{-1}(\Delta(0, 1))$ then $A(g) \cdot A(f^{-1}) = A(g \circ f^{-1}) = K(\alpha, \beta) \cdot K^{-1}(\alpha', \beta')$ which implies

$$K(\alpha, \beta)K^{-1}(\alpha', \beta') = K\left(\frac{1 - \beta'}{1 - \alpha' - \beta' + \beta\alpha'}, \frac{\beta - \beta'}{1 - \alpha' - \beta'}\right).$$

Sledd [6] proved $K(\alpha^*, \beta^*)$ is regular if and only if $\alpha^* = \beta^* = 0$ or $1 - |\alpha^*|^2 > (1 - \alpha^*)(1 - \beta^*) > 0$. Thus $K(\alpha', \beta')$ is regular if and only if (ii) holds and $K(\alpha, \beta) \cdot K^{-1}(\alpha', \beta')$ is regular if and only if (iv) holds.

Finally, $\Delta D_{\delta} \supseteq f^{-1}(\Delta(0, 1))$ if and only if

$$1 - |\alpha'|^2 - (1 - \alpha')(1 - \beta') + |(1 - \alpha')(1 - \beta')| < \frac{1}{|\beta'|},$$

since $f^{-1}(\Delta(0, 1))$ is a disk with center,

$$C = \frac{1 - |\alpha'|^2 - (1 - \alpha')(1 - \beta')}{|\beta'|^2 - |1 - \alpha' - \beta'|^2}$$

and radius

$$R = \frac{|1 - \beta'| |1 - \alpha'|}{|\beta'|^2 - |1 - \alpha' - \beta'|^2}$$

and (1) is equivalent to $|C| + |R| < 1/|\beta|$. Thus the transformed disk is contained in $D_\delta = \{z: |z| < 1/|\beta|\}$ if and only if (1) holds. Thus if (ii) holds (iii) is equivalent to (1).

The following corollaries to Theorem 4 give necessary and sufficient conditions for inclusion of some well-known matrix transforms by other matrix methods. In particular they answer some questions posed by Schoonmaker [5].
Corollary 1. If \( \frac{1}{2} < p \leq 1 \) and \( 0 < r < 1 \), then \( E(p) \subseteq T(r) \) if and only if \( \frac{(1-p)}{(2-p)} < r < \frac{p}{(2-p)} \).

Proof. With the notation of Theorem 4, \( E(p) = K(1-p, 0) \), \( T(r) = K(0, r) \). Thus conditions (i) and (ii) of that theorem are satisfied, because (i) is trivially true and (ii) is equivalent to \( 0 < p \leq 1 \). Condition (iii) with hypothesis \( \frac{1}{2} < p \leq 1 \) and \( 0 < r < 1 \) becomes
\[
| r | | 1 - | 1 - p |^2 | < | p |^2.
\]
This holds if and only if
\[
(1) \quad r < \frac{p}{(2 - p)}.
\]
Condition (iv) with \( \frac{1}{2} < p \leq 1 \) and \( 0 < r < 1 \) becomes
\[
| 1 - (1-p)(1-r) |^2 - | (1-p)(1-r) |^2 > (1-r)p > 0
\]
which is equivalent to
\[
(2) \quad r > \frac{(1-p)}{(2-p)}.
\]
But inequalities (1) and (2) can hold simultaneously only if \( \frac{1}{2} < p \) and thus the result follows.

Corollary 2. If \( 0 \leq r \leq \frac{1}{2} \) and \( | q | < 1 \) then \( T(r) \cap m \subseteq S(q) \) if and only if \( 0 < q < 1 - 2r \).

Proof. \( S(q) = K(1-q, q) \) and \( T(r) = K(0, r) \). If \( 0 \leq r \leq \frac{1}{2} \) and \( | q | < 1 \), condition (i) of Theorem 4 is satisfied, and (ii) is satisfied since (ii) is equivalent to regularity of \( T(r) \) or \( 0 \leq r < 1 \). Conditions (iii) and (iv) will be satisfied if and only if
\[
(1) \quad | q | < | 1 - r |^2 - | r |^2
\]
and
\[
(2) \quad | 1 - r |^2 - | (1-q)(1-r) |^2 > q(1-q)(1-r) > 0.
\]
It follows from \( q(1-q)(1-r) > 0 \) and \( | q | < 1 \) that \( q \) is real and \( q > 0 \). But then, under the hypothesis of the corollary, (1) becomes \( q < 1 - 2r \) and (2) becomes \( q > (2r-1)/r \). But this latter inequality is satisfied since \( (2r-1)/r < 0 < q \). Therefore (1) and (2) are equivalent to \( 0 < q < 1 - 2r \).

Corollary 3. If \( 0 \leq r \leq \frac{1}{3} \), then \( T(r) \cap m \subseteq E(p) \) if and only if \( 0 < p < 1 \).

Proof. \( K(0, r) = T(r) \) and \( K(1-p, 0) = E(p) \). If \( 0 \leq r \leq \frac{1}{3} \) then, a fortiori, \( 0 < r < 1 \) which is equivalent to condition (ii) and implies condition (i) of Theorem 4. Conditions (iii) and (iv) of that theorem are equivalent to
(1) \[ 0 < |r|^2 - |1 - r|^2, \]
and
(2) \[ |1 - r|^2 - |(1 - \rho)(1 - r)|^2 > \tilde{p}(1 - r) > 0, \]
respectively. (1) is trivially satisfied and (2) implies \( \rho \) is real, \( \rho > 0 \).

The first inequality in (2) thus reduces to \( \rho < 1 \) and the result follows.

Corollaries 1, 2, and 3 strengthen and add new results to theorems of Schoonmaker [5]. In conclusion it should be noted that results for \( E(\rho) \supset S(q) \) and \( T(\tau) \supset S(q) \) could not be found using the methods of this paper because \( f^{-1}(z) = \{z - (1 - q)/q^z \}, \ f(z) = (1 - q^z)/(1 - q^z) \), \( |q| < 1 \) is not analytic at the origin. This leads the author to suspect that \( S^{-1}(q) \) does not exist but no results along these lines could be found.

BIBLIOGRAPHY


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