ON A PROBLEM OF ERDÖS

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Erdös posed the following problem in [2, pp. 52–53]: Let $\{z_\nu\}$ be a sequence with $|z_\nu| = 1$ and put

$$B_k = \limsup_{m\to\infty} \left| \sum_{\nu=1}^{m} z_\nu^k \right|.$$

Is it true that $\limsup_{k\to\infty} B_k = \infty$?

In a subsequent paper [3] Erdös showed that there is an absolute constant C such that we always have $B_k/\log k > C$ infinitely often. He also conjected that $B_k/k > C$ for infinitely many values of k. Recently, Clunie [1] proved that

$$\limsup_{k\to\infty} B_k/k^{1/2} > 0,$$

and there is a sequence $\{z_\nu\}$ such that $B_k \leqq k$ $(k = 1, 2, \cdots)$.

In this paper we shall prove the following theorem.

THEOREM. *Let $\{z_\nu\}$ be any sequence with $|z_\nu| = 1$. If $\{z_\nu\}$ contains only a finite number of distinct elements, then for any $\delta > 0$,*

$$\limsup_{k\to\infty} B_k/k^{1-\delta} > \tfrac{1}{5}.$$

Let $z_\nu = \exp[i\phi_\nu]$ $(0 \leqq \phi_\nu < 2\pi)$. It is easy to see that for any sequence $\{z_\nu\}$ containing finite distinct elements there exists some $\epsilon > 0$ which depends on the sequence $\{z_\nu\}$ only, such that either

$$\phi_\nu - \phi_{\nu'} = 0 \quad \text{or} \quad |\phi_\nu - \phi_{\nu'}| \geqq \epsilon > 0$$

for all positive integers ν, ν'. So in what follows we shall use ϵ to denote the maximum positive constant satisfying the above condition.

In order to prove our Theorem we need two lemmas.

LEMMA 1. *Let $\{\zeta_\nu\}$ be any sequence containing only a finite number of distinct elements with $|\zeta_\nu| = 1$ $(\nu = 1, 2, \cdots)$. Let n, k be any positive integers and put*

$$S_k = \sum_{\nu=1}^{n} \zeta_\nu^k.$$

Then

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$$\frac{nr^2}{(1-r^2)^2} - \frac{n^2}{(1-\cos^2\epsilon)^2} \leqq \sum_{k=1}^{\infty} |S_k|^2 k r^{2k} \qquad (0<r<1).$$

PROOF. Let $z = re^{i\theta}$ $(0<r<1)$,

$$P_n(z) = \sum_{\nu=1}^{n}\sum_{k=1}^{\infty} k^{1/2}(\zeta_\nu z)^k = \sum_{k=1}^{\infty} k^{1/2} S_k z^k.$$

We see that

$$\begin{aligned}
\int_0^{2\pi} |P_n(z)|^2 d\theta &= \int_0^{2\pi} \sum_{\nu,\nu'=1}^{n} \sum_{k,k'=1}^{\infty} (kk')^{1/2} \zeta_\nu \bar{\zeta}_{\nu'} r^{k+k'} e^{i\theta(k-k')} d\theta \\
&= 2\pi \sum_{\nu,\nu'=1}^{n} \sum_{k=1}^{\infty} k(\zeta_\nu \bar{\zeta}_{\nu'} r^2)^k \\
&= 2\pi \sum_{\nu,\nu'=1}^{n} \frac{\zeta_\nu \bar{\zeta}_{\nu'} r^2}{(1-\zeta_\nu \bar{\zeta}_{\nu'} r^2)^2} \\
&= \frac{2\pi n r^2}{(1-r^2)^2} + 2\pi \sum_{\nu\neq\nu';\, 1\leq\nu,\nu'\leq n} \frac{\zeta_\nu \bar{\zeta}_{\nu'} r^2}{(1-\zeta_\nu \bar{\zeta}_{\nu'} r^2)^2},
\end{aligned} \tag{1}$$

and for $\phi_\nu \neq \phi_{\nu'}$

$$\begin{aligned}
\left|\frac{\zeta_\nu \bar{\zeta}_{\nu'} r^2}{(1-\zeta_\nu \bar{\zeta}_{\nu'} r^2)^2}\right| &\leqq \frac{1}{\inf\limits_{0<r<1} |1-\zeta_\nu \bar{\zeta}_{\nu'} r^2|^2} \\
&= \frac{1}{(1-\cos^2\epsilon)^2}.
\end{aligned} \tag{2}$$

On the other hand, we have

$$\int_0^{2\pi} |P_n(z)|^2 d\theta = 2\pi \sum_{k=1}^{\infty} |S_k|^2 k r^{2k}. \tag{3}$$

It follows from (1), (2), (3) that

$$\sum_{k=1}^{\infty} |S_k|^2 k r^{2k} \geqq \frac{nr^2}{(1-r^2)^2} - \frac{n^2}{(1-\cos^2\epsilon)^2} \qquad (0<r<1).$$

The proof of Lemma 1 is completed.

LEMMA 2. *Let* $\{z_\nu\}$ *be any sequence of complex numbers. If* $\limsup_{k\to\infty} B_k/k^{1-\delta} < C$, *for some absolute constants* δ, $C>0$, *then there is an integer* $k_1>0$ *which depends on* $\{z_\nu\}$, C *and* δ *only, such that given integers* n *and* $\mu(n)$, *with* $\mu(n) \geqq k_1$, *we can find* n *members of* $\{z_\nu\}$: ζ_1, $\zeta_2, \cdots, \zeta_n$ *satisfying*

$$|S_k| = \left|\sum_{\nu=1}^{n} \zeta_\nu^k\right| < 2Ck^{1-\delta} \qquad (k_1 \leqq k \leqq \mu(n)).$$

PROOF. From the given hypothesis there is an integer $k_1>0$ such that

$$B_k < Ck^{1-\delta} \qquad (k_1 \leqq k).$$

Since $B_k = \limsup_{m\to\infty} \left|\sum_{\nu=1}^{m} z_\nu^k\right|$, we see that for each $k \geqq k_1$, there exists an integer $N(k)>0$ such that

$$\left|\sum_{\nu=1}^{m} z_\nu^k\right| < Ck^{1-\delta} \qquad (m > N(k)).$$

For given n choose an integer $N>0$ so that

$$N \geqq \max_{k_1 \leq k \leq \mu(n)} \{N(k)\}.$$

Putting $\zeta_\nu = z_{N+\nu}$ $(\nu = 1, 2, \cdots, n)$, we have

$$\begin{aligned} |S_k| &= |\zeta_1^k + \zeta_2^k + \cdots + \zeta_n^k| \\ &= |z_{N+1}^k + z_{N+2}^k + \cdots + z_{N+n}^k| \\ &= \left|\sum_{\nu=1}^{N+n} z_\nu^k - \sum_{\nu=1}^{N} z_\nu^k\right| \\ &\leqq \left|\sum_{\nu=1}^{N+n} z_\nu^k\right| + \left|\sum_{\nu=1}^{N} z_\nu^k\right| \\ &< 2Ck^{1-\delta} \qquad (k_1 \leqq k \leqq \mu(n)). \end{aligned}$$

This completes the proof of Lemma 2.

We come now to prove our Theorem. For the given sequence $\{z_\nu\}$, suppose $\limsup_{k\to\infty} B_k/k^{1-\delta} < C$. By virtue of Lemma 2, there exists a positive integer k_1 such that for any given n and $\mu(n)$, with $\mu(n) \geqq k_1$, there are $\zeta_1, \zeta_2, \cdots, \zeta_n \in \{z_\nu\}$ with

$$|S_k| < 2Ck^{1-\delta} \qquad (k_1 \leqq k \leqq \mu(n)). \tag{4}$$

From Lemma 1 we have

$$\begin{aligned} \frac{-n^2}{(1-\cos^2\epsilon)^2} + \frac{nr^2}{(1-r^2)^2} &\leqq \sum_{k=1}^{\infty} |S_k|^2 k r^{2k} \\ &= \left(\sum_{k=1}^{k_1} + \sum_{k>k_1}^{\mu(n)} + \sum_{k>\mu(n)}^{\infty}\right) |S_k|^2 k r^{2k} \\ &= T_1 + T_2 + T_3, \text{ say.} \end{aligned} \tag{5}$$

Since $|S_k| \leqq n$, we have

$$T_1 = \sum_{k=1}^{k_1} |S_k|^2 k r^{2k} \leqq n^2 \sum_{k=1}^{k_1} k < n^2 k_1^2,$$

$$T_3 = \sum_{k>\mu(n)}^{\infty} |S_k|^2 k r^{2k} \leqq n^2 \sum_{k>\mu(n)}^{\infty} k r^{2k} \tag{6}$$

$$\leqq n^2 \mu(n) \frac{r^{2\mu(n)}}{(1-r^2)^2}.$$

Let A_k^α be the Cesàro numbers of order α $(0<\alpha)$, we have

$$\sum_{k=0}^{\infty} A_k^\alpha r^{2k} = \frac{1}{(1-r^2)^{1+\alpha}} \qquad (0 < r < 1),$$

$$A_k^\alpha = \frac{k^\alpha}{\Gamma(1+\alpha)} \{1 + o(1)\} \quad \text{as } k \to \infty.$$

See [4, Vol. I, pp. 76–77]. Assume that k_1 in (4) is large such that for every $k \geqq k_1$ we have $A_k^{3-\delta}\Gamma(4) \geqq k^{3-\delta}$. From (4) it follows that

$$T_2 = \sum_{k>k_1}^{\mu(n)} |S_k|^2 k r^{2k} \leqq 4C^2 \sum_{k>k_1}^{\mu(n)} k^{3-\delta} r^{2k} \tag{7}$$

$$\leqq 4C^2 \Gamma(4) \sum_{k=0}^{\infty} A_k^{3-\delta} r^{2k} = 24C^2 \frac{1}{(1-r^2)^{4-\delta}}.$$

From (5), (6), (7) we have

$$\frac{-n^2}{(1-\cos^2\epsilon)^2} + \frac{nr^2}{(1-r^2)^2} \leqq n^2 k_1^2 + \frac{24C^2}{(1-r^2)^{4-\delta}} + \frac{n^2\mu(n) r^{2\mu(n)}}{(1-r^2)^2}. \tag{8}$$

Put $1/(1-r^2) = n^{1/(2-\delta)}$ and $\mu(n) = n^2$. Then we have

$$\frac{n}{(1-r^2)^2} = \frac{1}{(1-r^2)^{4-\delta}} = n^{2+(\delta/(2-\delta))},$$

and

$$\frac{n^2\mu(n) r^{2\mu(n)}}{(1-r^2)^2} < n^6 (1 - n^{-1/(2-\delta)})^{n^2} < n^6 e^{-n} = o(1),$$

as $n \to \infty$. Thus, taking $n \to \infty$, from (8) we have

$$1 \leqq 24C^2.$$

This implies that

$$\limsup_{k\to\infty} B_k/k^{1-\delta} \geqq 1/\sqrt{24} > 1/5 \qquad (\delta > 0).$$

The proof is completed.

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References

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