ON A PROBLEM OF ERDŐS

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Erdös posed the following problem in [2, pp. 52-53]: Let \( \{z_r\} \) be a sequence with \( |z_r| = 1 \) and put

\[
B_k = \limsup_{m \to \infty} \left| \sum_{r=1}^{m} z_r^k \right|
\]

Is it true that \( \limsup_{k \to \infty} B_k = \infty \)?

In a subsequent paper [3] Erdös showed that there is an absolute constant \( C \) such that we always have \( B_k/\log k > C \) infinitely often. He also conjectured that \( B_k/k > C \) for infinitely many values of \( k \). Recently, Clunie [1] proved that

\[
\limsup_{k \to \infty} B_k/k^{1/2} > 0,
\]

and there is a sequence \( \{z_r\} \) such that \( B_k \leq k \) (\( k = 1, 2, \cdots \)).

In this paper we shall prove the following theorem.

**Theorem.** Let \( \{z_r\} \) be any sequence with \( |z_r| = 1 \). If \( \{z_r\} \) contains only a finite number of distinct elements, then for any \( \delta > 0 \),

\[
\limsup_{k \to \infty} B_k/k^{1-\delta} > \frac{1}{\delta}.
\]

Let \( z_r = \exp[i\phi_r] \) (\( 0 \leq \phi_r < 2\pi \)). It is easy to see that for any sequence \( \{z_r\} \) containing finite distinct elements there exists some \( \epsilon > 0 \) which depends on the sequence \( \{z_r\} \) only, such that either

\[
\phi_r - \phi_r' = 0 \quad \text{or} \quad |\phi_r - \phi_r'| \geq \epsilon > 0
\]

for all positive integers \( r, r' \). So in what follows we shall use \( \epsilon \) to denote the maximum positive constant satisfying the above condition.

In order to prove our Theorem we need two lemmas.

**Lemma 1.** Let \( \{\xi_r\} \) be any sequence containing only a finite number of distinct elements with \( |\xi_r| = 1 \) (\( \nu = 1, 2, \cdots \)). Let \( n, k \) be any positive integers and put

\[
S_k = \sum_{r=1}^{n} \xi_r^k.
\]

Then

Received by the editors August 12, 1968.
\[
\frac{n r^2}{(1 - r^2)^2} - \frac{n^2}{(1 - \cos^2 \phi)^2} \leq \sum_{k=1}^{\infty} |S_k| r^{2k} < 1.
\]

**Proof.** Let \( z = re^{i\theta} \) \((0 < r < 1)\),
\[
P_n(z) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k^{1/2}(\xi z)^k = \sum_{k=1}^{\infty} k^{1/2}S_k z^k.
\]

We see that
\[
\int_0^{2\pi} |P_n(z)|^2 d\theta = \int_0^{2\pi} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (k k')^{1/2} \xi z \xi z' r^{k+k'} e^{i(k-k')} d\theta
\]
\[
= 2\pi \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k \xi z \xi z' r^2
\]
\[
= 2\pi \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \xi z \xi z' r^2
\]
\[
= 2\pi n r^2 + 2\pi \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \xi z \xi z' r^2
\]
and for \( \phi, \phi' \),
\[
\inf_{0 < r < 1} |1 - \xi z \xi z'|^2 \leq \frac{1}{\inf_{0 < r < 1} |1 - \xi z \xi z'|^2}
\]
\[
= \frac{1}{\cos \phi \cos \phi'}.
\]

On the other hand, we have
\[
\int_0^{2\pi} |P_n(z)|^2 d\theta = 2\pi \sum_{k=1}^{\infty} |S_k| r^{2k}.
\]

It follows from (1), (2), (3) that
\[
\sum_{k=1}^{\infty} |S_k| r^{2k} \geq \frac{n r^2}{(1 - r^2)^2} - \frac{n^2}{(1 - \cos^2 \phi)^2} \quad (0 < r < 1).
\]

The proof of Lemma 1 is completed.

**Lemma 2.** Let \( \{z_n\} \) be any sequence of complex numbers. If
\[
\limsup_{k \to \infty} B_k/k^{1-\delta} < C, \text{ for some absolute constants } \delta, C > 0,
\]
then there is an integer \( k_1 > 0 \) which depends on \( \{z_n\} \), \( C \) and \( \delta \) only, such that given
\[
\text{integers } n \text{ and } \mu(n), \text{ with } \mu(n) \geq k_1, \text{ we can find } n \text{ members of } \{z_n\} : \xi_1, \xi_2, \ldots, \xi_n \text{ satisfying}
\]
\[ |S_k| = \left| \sum_{r=1}^{n} \xi_r \right| < 2Ck^{1-\delta} \quad (k_1 \leq k \leq \mu(n)). \]

**Proof.** From the given hypothesis there is an integer \( k_1 > 0 \) such that
\[ B_k < Ck^{1-\delta} \quad (k_1 \leq k). \]
Since \( B_k = \limsup_{m \to \infty} \left| \sum_{r=1}^{m} a_r \right| \), we see that for each \( k \geq k_1 \), there exists an integer \( N(k) > 0 \) such that
\[ \left| \sum_{r=1}^{m} a_r \right| < Ck^{1-\delta} \quad (m > N(k)). \]
For given \( n \) choose an integer \( N > 0 \) so that
\[ N \geq \max_{k_1 \leq k \leq \mu(n)} \{ N(k) \}. \]
Putting \( \xi_r = z_{n+r} \ (r = 1, 2, \ldots, n) \), we have
\[ |S_k| = |\xi_1 + \xi_2 + \cdots + \xi_n| = \left| \sum_{r=1}^{N+n} \xi_r - \sum_{r=1}^{N} \xi_r \right| \leq \left| \sum_{r=1}^{N+n} \xi_r \right| + \left| \sum_{r=1}^{N} \xi_r \right| < 2Ck^{1-\delta} \quad (k_1 \leq k \leq \mu(n)). \]
This completes the proof of Lemma 2.

We come now to prove our Theorem. For the given sequence \( \{z_r\} \), suppose \( \limsup_{n \to \infty} B_k/k^{1-\delta} < C \). By virtue of Lemma 2, there exists a positive integer \( k_1 \) such that for any given \( n \) and \( \mu(n) \), with \( \mu(n) \geq k_1 \), there are \( \xi_1, \xi_2, \ldots, \xi_n \in \{z_r\} \) with
\[ |S_k| < 2Ck^{1-\delta} \quad (k_1 \leq k \leq \mu(n)). \]
From Lemma 1 we have
\[ \frac{-\pi^2}{(1 - \cos^2 \theta)} + \frac{\pi^2}{(1 - r^2)^2} \leq \sum_{k=1}^{\infty} \left| S_k \right|^2k^{2k} \]
(4) \[ = \left( \sum_{k=1}^{k_1} + \sum_{k > k_1} + \sum_{k > \mu(n)} \right) \left| S_k \right|^2k^{2k} \]
(5) \[ = T_1 + T_2 + T_3, \text{ say.} \]
Since $|S_k| \leq n$, we have

$$T_1 = \sum_{k=1}^{k_1} |S_k|^2 k r^{2k} \leq n^2 \sum_{k=1}^{k_1} k < n^2 k_1,$$

(6)

$$T_2 = \sum_{k>p(n)} \sum_{k'=r^2k} |S_k|^2 k r^{2k} \leq n^2 \sum_{k>p(n)} k r^{2k} \leq n^2 \mu(n) \frac{r^{2\mu(n)}}{(1 - r^2)^2}.$$

Let $A^\alpha_k$ be the Cesàro numbers of order $\alpha$ ($0 < \alpha$), we have

$$\sum_{k=0}^{\infty} A^{2\alpha}_k = \frac{1}{(1 - r^2)^{1+\alpha}} \quad (0 < r < 1),$$

$$A^\alpha_k = \frac{k^n}{\Gamma(1 + \alpha)} \{1 + o(1)\} \quad \text{as } k \to \infty.$$

See [4, Vol. I, pp. 76–77]. Assume that $k_1$ in (4) is large such that for every $k \geq k_1$ we have $A^{3-4}_k \Gamma(4) \geq k^{3-4}$. From (4) it follows that

$$T_2 = \sum_{k>p(n)} \sum_{k'=r^{2k}} |S_k|^2 k r^{2k} \leq 4C^2 \sum_{k>p(n)} k^{3-4} r^{2k} \leq 4C^2 \Gamma(4) \sum_{k=0}^{\infty} A^{3-4}_k r^k = 24C^2 \frac{1}{(1 - r^2)^{4-5}}.$$

(7)

From (5), (6), (7) we have

$$\frac{-n^2}{(1 - \cos^2 e)^2} + \frac{nr^2}{(1 - r^2)^2} \leq n^2 k_1 + \frac{24C^2}{(1 - r^2)^{4-5}} + \frac{n^2 \mu(n) r^{2\mu(n)}}{(1 - r^2)^2}.$$

Put $1/(1 - r^2) = n^{1/(2-\delta)}$ and $\mu(n) = n^2$. Then we have

$$\frac{n}{(1 - r^2)^2} = \frac{1}{(1 - r^2)^{4-5}} = n^{2 + (\delta/(2-\delta))},$$

and

$$\frac{n^2 \mu(n) r^{2\mu(n)}}{(1 - r^2)^2} < n^6 (1 - n^{-1/(2-\delta)})^{-1} < n^6 e^{-n} = o(1),$$

as $n \to \infty$. Thus, taking $n \to \infty$, from (8) we have

$$1 \leq 24C^2.$$

This implies that
\[
\lim_{k \to \infty} \sup B_k/k^{1-\delta} \geq 1/\sqrt{24} > 1/5 \quad (\delta > 0).
\]

The proof is completed.

My thanks are due to the referee for some useful remarks and improvements in this paper.

References


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