ON THE ENTROPY OF CERTAIN CLASSES OF SKEW-PRODUCT TRANSFORMATIONS

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1. **Introduction.** For the theory of Lebesgue spaces, skew-product transformations and entropy see [5], [4] and [6]. Let X and Y be Lebesgue spaces with measures μ and ν and let $M=X\times Y$ be the direct product space with the direct product measure $\mu\times\nu$. Let S be an automorphism of the space X and let $\{T_x\}$ be a family of automorphisms of the space Y, define $U(x, y)=(Sx, T_xy)$. Under certain measurability conditions on the family $\{T_x\}$ U is an automorphism of the space M. U is called the skew-product with base S and fibre T_x. The entropy of such a skew-product has been calculated [1], [4], but the formulae given are not in general amenable to calculation.

We will consider two special types of skew-product:

1. Let T be an automorphism of Y and let $k(x)$ be an integer valued function on X such that $k(x)>0$ and $\int_X k(x)d\mu<\infty$. The first class of skew-products are those for which $T_x=T^{k(x)}$.

2. Let ψ_t be a measurable measure-preserving flow on Y and let $f(x)$ be a real-valued function on X such that $f(x)\geqq\tau>0$, for some constant τ, and $\int_X f(x)d\mu<\infty$. The second class of skew-products are those of the form $T_x=\psi_{f(x)}$.

In both these special cases we will show that $h(U) = h(S) +\int_X h(T_x)d\mu$. We note that the first class is not contained in the second since the automorphism T may not be embeddable in a flow.

2. **Induced automorphisms and special flows.** Let T be an automorphism of a Lebesgue space M with measure μ. If $X\subset M$ and $\bigcup_{n=0}^{\infty} T^nX=M$, then T induces in the Lebesgue space X, with measure $\mu_X(A)=\mu(A)(\mu(X))^{-1}$, $A\subset X$, the induced automorphism T' given by $T'x=T^{k(x)}x$ $(x\in X)$ where $k(x)$ is the smallest positive integer l such that $T^lx\in X$. Abramov [2] has shown that $h(T)=h(T')\mu(X)$.

Let us now consider the special flow built on an automorphism S of a Lebesgue space M under a function f. f is a real-valued function on M such that

(i) $f(x)\geqq\tau>0$ for $x\in M$,

(ii) $\int_M f(x)d\mu<\infty$.

Let V be the subset of $M\times R$, R is real line, defined by $(x, u)\in V$ if $0\leqq u<f(x)$. We define a group of automorphisms on V as follows; for $t<\tau$

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$$\begin{aligned} S_t(x, u) &= (x, u + t) \qquad \text{if } t < f(x) - u, \\ &= (Sx, t + u - f(x)) \qquad \text{if } t \geqq f(x) - u, \end{aligned}$$

for the remaining values of t the automorphisms $\{S_t\}$ can be defined so that $\{S_t\}$ is a group. For an explicit formula for S_t see [6]. Abramov [3] has shown that $h(S_t) = |t| h(S)(\int_M f(x)d\mu)^{-1}$. We note that the entropy of an automorphism is always calculated with respect to a normalised measure space.

3. Main theorems.

THEOREM 1. *Let U on $X \times Y$ with measure $\mu \times \nu$ be defined by $U(x, y) = (Sx, T^{k(x)}y)$ where $\infty > k(x) > 0$, $k(x)$ is integer valued and $\int_X k(x)d\mu < \infty$. Then*

$$h(U) = h(S) + h(T)\int_X k(x)d\mu = h(S) + \int_X h(T^{k(x)})d\mu.$$

PROOF. Let Z denote the set of positive integers $Z = \{1, 2, \cdots\}$ and let η be the measure on Z which assigns measure 1 to each point. Let V be the subset of $X \times Y \times Z$ defined by $(x, y, i) \in V$ if $i \leqq k(x)$. Note that $V = V' \times Y$ where V' is the subset of $X \times Z$ defined by $(x, i) \in V'$ if $i \leqq k(x)$. It is easily seen that

$$\mu \times \nu \times \eta(V) = \mu \times \eta(V') = \int_X k(x)d\mu.$$

Let us consider V as a Lebesgue space with the measure normalised, i.e., if $A \subset V$ then

$$\mu'(A) = \mu \times \nu \times \eta(A) \cdot \left(\int_X k(x)d\mu\right)^{-1}.$$

Define an automorphism ϕ on V by

$$\begin{aligned} \phi(x, y, i) &= (x, Ty, i + 1) \qquad \text{if } i < k(x), \\ &= (Sx, Ty, 1) \qquad \text{if } i = k(x). \end{aligned}$$

Then $\phi = T \times \psi$ where ψ is the automorphism on V' defined by

$$\begin{aligned} \psi(x, i) &= (x, i + 1) \qquad \text{if } i < k(x). \\ &= (Sx, 1) \qquad \text{if } i = k(x). \end{aligned}$$

It is clear that U is the automorphism induced by ϕ on the set $X \times Y \times \{1\}$. Thus we have

$$h(U) = h(\phi)(\mu'(X \times Y \times \{1\}))^{-1} = \int_X k(x)d\mu h(\phi).$$

Since $\phi = T \times \psi$ it follows that $h(\phi) = h(T) + h(\psi)$ and it remains only to calculate $h(\psi)$. It is clear that S is the automorphism induced by ψ on the set $X \times \{1\}$; thus $h(\psi) = h(S) \cdot (\int_X k(x) d\mu)^{-1}$. Finally

$$h(U) = h(S) + \int_X k(x) d\mu \cdot h(T)$$

$$= h(S) + \int_X h(T^{k(x)}) d\mu.$$

THEOREM 2. *Let U on $X \times Y$ with measure $\mu \times \nu$ be defined by $U(x, y) = (Sx, \psi_{f(x)} y)$ where $f(x)$ is a real-valued function on X, $\infty > f(x) \geqq \tau > 0$ such that $\int_X f(x) d\mu < \infty$. Then*

$$h(U) = h(S) + \int_X f(x) d\mu h(\psi_1) = h(S) + \int_X h(\psi_{f(x)}) d\mu.$$

PROOF. Let R be the set of real numbers with the usual Lebesgue measure. Let A be the subset of $X \times Y \times R$ defined by $(x, y, u) \in A$ if $0 \leqq u < f(x)$ we note that $A = A' \times Y$ where A' is the subset of $X \times R$ defined by $(x, u) \in A'$ if $0 \leqq u < f(x)$. We consider the following flows defined on A, A and A' respectively, all are defined for $t < \tau$ and extended by group property,

$$\begin{aligned}
\phi_t(x, y, u) &= (x, \psi_t y, u + t) \qquad \text{if } t < f(x) - u, \\
&= (Sx, \psi_t y, u + t - f(x)) \qquad \text{if } t \geqq f(x) - u; \\
\phi_t'(x, y, u) &= (x, y, u + t) \qquad \text{if } t < f(x) - u, \\
&= (Sx, \psi_{f(x)} y, u + t - f(x)) \qquad \text{if } t \geqq f(x) - u; \\
\phi_t''(x, u) &= (x, u + t) \qquad \text{if } t < f(x) - u, \\
&= (Sx, u + t - f(x)) \qquad \text{if } t \geqq f(x) - u.
\end{aligned}$$

It is easy to see that ϕ_t is a direct product of the flow ψ_t on Y with the flow ϕ_t'' on A'. It is also clear that ϕ_t' and ϕ_t'' are in the form of special flows built over the automorphisms U and S respectively and with the same function $f(x)$, regarded firstly as a function of two variables $f(x, y) \equiv f(x)$ and secondly just as a function of one variable. Using Abramov's formula

$$h(\phi_t') = |t| \, h(U) \left(\int_X f(x) d\mu \right)^{-1},$$

$$h(\phi_t'') = |t| \, h(S) \left(\int_X f(x) d\mu \right)^{-1}.$$

Also $h(\phi_t) = h(\psi_t) + h(\phi_t'')$. We will exhibit an isomorphism U of A

to itself which has the property that $U^{-1}\phi_t U = \phi_t'$ for all t. Define

$$U(x, y, u) = (x, \psi_u y, u);$$

then, it is easily checked that U satisfies the required property. Thus $h(\phi_t) = h(\phi_t')$. Gathering together the formulae we have derived we obtain

$$h(\phi_t) = |t|\, h(U)\left(\int_X f(x)d\mu\right)^{-1} = h(\psi_t) + |t|\, h(S)\left(\int_X f(x)d\mu\right)^{-1}$$

or, in other words,

$$\begin{aligned} h(U) &= h(S) + h(\psi_t)\,|t|^{-1}\int_X f(x)d\mu \\ &= h(S) + h(\psi_1)\int_X f(x)d\mu, \end{aligned}$$

since $h(\psi_t) = |t|\, h(\psi_1)$.

The proof of Theorem 1 depends on the restriction that $0 < k(x)$ and a natural question is to ask if the theorem is true when we let $k(x)$ take both positive and negative values. The following partial result was pointed out to us by W. Parry.

THEOREM 3. *Let $h(x)$ be an integer valued function on X with the following properties*

(i) $\int_X h(x)d\mu < \infty$.

(ii) *There exists an integer-valued function $k(x)$ and an integrable integer-valued function $l(x)$ such that $h(x) = k(x) - l(x) + l(Sx)$ and $0 < k(x) < \infty$.*

Then, if $U(x, y) = (Sx, T^{h(x)}y)$, $h(U) = h(S) + h(T)\int_X h(x)d\mu$.

We remark that it is possible to find functions $h(x)$ which take both positive and negative values and in such cases

$$h(T)\int_X h(x)d\mu \neq \int_X h(T^{h(x)})d\mu.$$

PROOF. Let $\phi(x, y) = (x, T^{l(x)}y)$; then

$$\begin{aligned} \phi^{-1}U\phi(x, y) &= (Sx, T^{-l(Sx)}T^{h(x)}T^{l(x)}y) \\ &= (Sx, T^{k(x)}y). \end{aligned}$$

By Theorem 1,

$$h(U) = h(\phi^{-1}U\phi) = h(S) + h(T)\int_X k(x)d\mu.$$

But $h(x)=k(x)-l(x)+l(Sx)$; therefore, since S preserves the measure μ,

$$\int_X h(x)d\mu = \int_X k(x)d\mu.$$

Thus $h(U)=h(S)+h(T)\int_X h(x)d\mu$. A theorem of a similar type will also extend Theorem 2.

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