FINITELY GENERATED CLASSES OF SETS OF NATURAL NUMBERS

JULIA ROBINSON

We say that a set S of natural numbers is generated by a class $\mathfrak{F}$ of functions if S is the range of a function F obtained by composition from the functions of $\mathfrak{F}$. Here we consider the identity function I to be obtained by the empty composition. We consider only the case in which $\mathfrak{F}$ consists of functions of one variable on and to the set of natural numbers $\boldsymbol{N}$. All sets considered are subsets of $\boldsymbol{N}$.

A class $\mathfrak{C}$ of sets is generated by $\mathfrak{F}$ if every nonempty set of $\mathfrak{C}$ is generated by $\mathfrak{F}$ and every set generated by $\mathfrak{F}$ is in $\mathfrak{C}$. $\mathfrak{C}$ is finitely generated if there is a finite class of functions $\mathfrak{F}$ which generates $\mathfrak{C}$.

A function F is compatible with $\mathfrak{C}$ if $F(\boldsymbol{S})$ belongs to $\mathfrak{C}$ for every $\boldsymbol{S}$ in $\mathfrak{C}$.

Clearly, if $\mathfrak{F}$ generates $\mathfrak{C}$ then every F in $\mathfrak{F}$ is compatible with $\mathfrak{C}$. Also $\boldsymbol{N}$ belongs to $\mathfrak{C}$. Furthermore to show that $\mathfrak{F}$ generates $\mathfrak{C}$ it is sufficient to prove that every $F \in \mathfrak{F}$ is compatible with $\mathfrak{C}$, $\boldsymbol{N} \in \mathfrak{C}$, and every nonempty set in $\mathfrak{C}$ is generated by $\mathfrak{F}$.

In [1] finite sets of relatively simple functions are given which generate the classes of recursively enumerable sets and diophantine sets. Here we ask: What classes of sets can be finitely generated? In particular, we will show that if $\mathfrak{C}$ is a denumerable field of sets such that all finite sets belong to $\mathfrak{C}$ and at least one infinite set having an infinite complement belongs to $\mathfrak{C}$, then $\mathfrak{C}$ is finitely generated.

LEMMA 1. *If $\mathfrak{I}$ is an infinite class of infinite sets then there is a set $\boldsymbol{M}$ such that $\boldsymbol{M} \cap \boldsymbol{T}$ and $\overline{\boldsymbol{M}} \cap \boldsymbol{T}$ are infinite for every $\boldsymbol{T}$ belonging to $\mathfrak{I}$.*

PROOF. Let each $\boldsymbol{T}$ in $\mathfrak{I}$ be listed infinitely often in the sequence $\boldsymbol{T}_0, \boldsymbol{T}_1, \cdots$. We will construct $\boldsymbol{M}$ and $\overline{\boldsymbol{M}}$ simultaneously. At the nth step put the least unassigned number of $\boldsymbol{T}_n$ in $\boldsymbol{M}$ and then the least remaining unassigned number of $\boldsymbol{T}_n$ in $\overline{\boldsymbol{M}}$.

LEMMA 2. *Given infinite sets $\boldsymbol{T}_0, \boldsymbol{T}_1, \cdots$ and nonempty sets $\boldsymbol{S}_0, \boldsymbol{S}_1, \cdots$ such that $\boldsymbol{S}_k = \boldsymbol{S}_l$ whenever $\boldsymbol{T}_k \cap \boldsymbol{T}_l$ is not empty. Then there is a function F such that $F(\boldsymbol{T}_n) = \boldsymbol{S}_n$.*

PROOF. Make a list of natural numbers such that each number occurs infinitely often. We will use the list to define F at one argument for each step of the construction. Namely, if at a given step n is the

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next number in the list and if x is the least unassigned number in $\boldsymbol{T}_n$ and y is the least number in $\boldsymbol{S}_n$ not yet in $F(\boldsymbol{T}_n)$, put $Fx=y$; while if there is no such y, put $Fx=$ the least number of $\boldsymbol{S}_n$. Finally, in case $\bigcup \boldsymbol{T}_k \neq \boldsymbol{N}$, put $Fx=0$ for every x not in any $\boldsymbol{T}_k$.

THEOREM 1. HYPOTHESES. (1) $\mathcal{C}$ *is a denumerable class of sets.* (2) *If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *belong to* $\mathcal{C}$ *then* $\boldsymbol{A}\cup\boldsymbol{B}$ *belongs to* $\mathcal{C}$. (3) $\boldsymbol{N}$ *belongs to* $\mathcal{C}$. (4) *Either* (i) $\mathcal{C}$ *contains all finite sets or* (ii) $\mathcal{C}$ *contains only infinite sets and* $\mathcal{C}$ *contains the set obtained from a member set by adjoining a single element.* (5) *There is an infinite partition* $\mathcal{P}\subset\mathcal{C}$ *of* $\boldsymbol{N}$ *into infinite sets all of which are generated by a finite class* $\mathfrak{F}$ *of functions compatible with* $\mathcal{C}$.

CONCLUSION. $\mathcal{C}$ *is finitely generated.*

PROOF. Let $\mathcal{P}=\{\boldsymbol{P}_0, \boldsymbol{P}_1, \cdots\}$. We say that $\boldsymbol{I}$ is the index set of a set $\boldsymbol{S}$ if $\boldsymbol{I}=\{n: \boldsymbol{S}\cap\boldsymbol{P}_n\neq\varnothing\}$. Let $\boldsymbol{M}$ be the set given by Lemma 1 such that every infinite index set of a set belonging to $\mathcal{C}$ contains infinitely many elements of both $\boldsymbol{M}$ and $\overline{\boldsymbol{M}}$. (Concerning $\overline{\boldsymbol{M}}$, we shall use only that $\overline{\boldsymbol{M}}$ is infinite.) List the nonempty sets of $\mathcal{C}$ in a sequence $\boldsymbol{T}_0$, $\boldsymbol{T}_1, \cdots$ so that $\boldsymbol{T}_m=\boldsymbol{N}$ for every m in $\boldsymbol{M}$. We will define F in such a way that (a) $F(\boldsymbol{T}_n\cap\boldsymbol{P}_k)=\boldsymbol{T}_k$ whenever $\boldsymbol{T}_n\cap\boldsymbol{P}_k$ is infinite, and (b) $F(\boldsymbol{T}_n)=\boldsymbol{N}$ whenever $\boldsymbol{T}_n$ has an infinite index set. Let us check that this is sufficient. In the first place, every nonempty set of $\mathcal{C}$ will be generated by $\{F\}\cup\mathfrak{F}$ since $F(\boldsymbol{P}_k)=\boldsymbol{T}_k$. Also F is compatible with $\mathcal{C}$, i.e. for every n, $F(\boldsymbol{T}_n)$ belongs to $\mathcal{C}$. To see this, let $\boldsymbol{I}$ be the index set of $\boldsymbol{T}_n$. If $\boldsymbol{I}$ is infinite then $F(\boldsymbol{T}_n)=\boldsymbol{N}$ and $\boldsymbol{N}$ belongs to $\mathcal{C}$ by (3). If $\boldsymbol{I}$ is finite, then

$$F(\boldsymbol{T}_n) = \bigcup_{k\in \boldsymbol{I}} F(\boldsymbol{T}_n\cap\boldsymbol{P}_k),$$

where $F(\boldsymbol{T}_n\cap\boldsymbol{P}_k)$ is either finite or $\boldsymbol{T}_k$ according as $\boldsymbol{T}_n\cap\boldsymbol{P}_k$ is finite or infinite. Hence if $\mathcal{C}$ contains all finite sets then $F(\boldsymbol{T}_n)$ is a nonempty finite union of sets of $\mathcal{C}$ and so $F(\boldsymbol{T}_n)$ belongs to $\mathcal{C}$. If $\mathcal{C}$ consists only of infinite sets, then $\boldsymbol{T}_n\cap\boldsymbol{P}_k$ is infinite for some k. Thus $F(\boldsymbol{T}_n)$ is the union of a nonempty finite union of sets of $\mathcal{C}$ and a finite set. Hence $F(\boldsymbol{T}_n)$ belongs to $\mathcal{C}$.

Therefore to complete the proof, we need only define F satisfying (a) and (b). Now make a list of pairs (n, k) so that each pair occurs infinitely often. We will use this list to index the steps in the definition of F. At a step corresponding to (n, k), if there is a number in $\boldsymbol{T}_n\cap\boldsymbol{P}_k$ at which F has not yet been assigned, let the least such number be x. Put Fx equal to the least number in $\boldsymbol{T}_k$ which is not already in $F(\boldsymbol{T}_n\cap\boldsymbol{P}_k)$ if there is such a number; otherwise put Fx equal to the least number in T_k. Also if the index set of T_n is infinite and y is the least number unassigned in $T_n \cap P_m$ for any m in M, define $Fy=k$. Then F will satisfy both (a) and (b).

EXAMPLES. The classes of recursive sets, recursively enumerable sets, arithmetical sets, hyperarithmetical sets, are all finitely generated. We can let $P_k=(SD)^k D(N)$ since S, D are compatible with each of these classes. Here S is the successor function and D the double function.

By the usual diagonal argument it is easy to see that the class of arithmetical sets cannot be generated by a finite class of arithmetical functions. (Similarly hyperarithmetical sets cannot be generated by a finite class of hyperarithmetical functions.) I do not know if the class of recursive sets can be generated by a finite number of recursive functions.

An interesting class of sets which is not covered by Theorem 1 is the class of polynomial sets and I do not know whether it is finitely generated. By a polynomial set, I mean the range of a polynomial with positive integer coefficients and any number of variables.

If we assume that $\mathcal{C}$ is a field of sets which contains all finite sets, then condition (5) in Theorem 1 can be weakened to the mere existence of $\mathcal{P}$. In order to prove this, we shall need the following combinatorial result.

LEMMA 3. *Let $\mathcal{C}$ be a denumerable field of sets which contains all finite sets. Suppose further that there is an infinite partition $\mathcal{P}_0$ of N into infinite sets of $\mathcal{C}$. Then there is an infinite partition $\mathcal{P}$ of N into infinite sets $P_0, P_1, \cdots$ of $\mathcal{C}$ such that for every S in $\mathcal{C}$ either S or $\overline{S}$ is included in a finite union of the P's.*

REMARK. This lemma is stronger than the one I originally proved and was suggested to me by Dana Scott. His proof was obtained by considering the Boolean algebra of sets of $\mathcal{C}$ mod finite sets.

PROOF. Let $S_0, S_1, \cdots$ be the sets of $\mathcal{C}$. We will define $P_0, P_1, \cdots$ by means of a sequence of infinite partitions $\mathcal{P}_0, \mathcal{P}_1, \cdots$ of N into infinite sets of $\mathcal{C}$ such that for every k and n with $k<n$, P_k belongs to $\mathcal{P}_n$ and either S_k or $\overline{S}_k$ is included in $\bigcup_{j<n} P_j$. Assume that these conditions hold for all n up through t. We shall define P_t and $\mathcal{P}_{t+1}$ so that they also hold for $n=t+1$. Let S be S_t if $\overline{S}_t$ has infinitely many infinite parts under $\mathcal{P}_t$; otherwise let S be $\overline{S}_t$. In either case, $\overline{S}$ has infinitely many infinite parts under $\mathcal{P}_t$. Let those infinite parts of $\overline{S}$ which are not contained in one of $P_0, \cdots, P_{t-1}$ be $I_0, I_1, \cdots$. Then put

$$P_t = I_0 \cup \left(S - \bigcup_{k<t} P_k\right).$$

Let $a_1, a_2, \cdots$ be the numbers which do not belong to

$$\bigcup_{k \leq t} P_k \cup \bigcup_{j \geq 1} I_j,$$

if there are any. Put $T_k = I_k \cup \{a_k\}$ if there is an a_k; otherwise let $T_k = I_k$. Finally, put

$$\mathcal{P}_{t+1} = \{P_0, P_1, \cdots, P_t, T_1, \cdots\}.$$

Clearly $P_t \in \mathcal{P}_{t+1}$, $S \subseteq \bigcup_{k \leq t} P_k$, and $\mathcal{P}_{t+1}$ is an infinite partition of N into infinite sets of $\mathcal{C}$.

Now let $\mathcal{P} = \{P_0, P_1, \cdots\}$. By construction each P_k is an infinite set of $\mathcal{C}$, $\mathcal{P}$ is disjoint, and for every k, either S_k or $\bar{S}_k$ is contained in $\bigcup_{j \leq k} P_j$. Finally, every n belongs to some P_j. In fact, $\{n\} = S_l$ for some l, so n belongs to some P_j with j not exceeding l. Hence $\mathcal{P}$ satisfies the lemma.

THEOREM 2. *If a denumerable field of sets $\mathcal{C}$ contains all finite sets and includes an infinite disjoint class $\mathcal{P}$ of infinite sets $P_0, P_1, \cdots$, then $\mathcal{C}$ is finitely generated.*

PROOF. Without loss of generality, we may suppose that $\mathcal{P}$ is a partition of N; otherwise adjoin the missing numbers to the parts P_k with at most one to each P_k. By Lemma 3, we may also assume that $\mathcal{P}$ has the further property that for every S in $\mathcal{C}$ either S or $\bar{S}$ is included in a finite union of P's.

By Lemma 2, we can define a function H so that $H(S) = P_0$ for every infinite set S in $\mathcal{C}$. Hence H is compatible with $\mathcal{C}$.

Let G be defined so that $G(S) = P_{n+1}$ for every infinite set S of $\mathcal{C}$ which is included in P_n. The existence of G also follows from Lemma 2. Then $G^k H(N) = P_k$ for all k. Furthermore, if S is included in a finite union of P's, then $G(S)$ is the union of a finite set and a finite union of P's; if $\bar{S}$ is included in a finite union of P's, then $G(S)$ is the union of a finite set and the complement of a finite union of P's. In both cases $G(S)$ belongs to $\mathcal{C}$. Hence G is compatible with $\mathcal{C}$.

We can now finish the proof by noticing the hypotheses of Theorem 1 hold. We can also construct F more directly as follows: Let S_0, $S_1, \cdots$ be the nonempty sets of $\mathcal{C}$. By Lemma 2, there is a function F such that $F(T) = S_k$ if T is an infinite subset of P_k which belongs to $\mathcal{C}$. Then F is compatible with $\mathcal{C}$. Indeed, if S is contained in a finite union of P's then $F(S)$ is the union of a finite number of sets of $\mathcal{C}$ and if $\bar{S}$ is contained in a finite union of P's then $F(S) = N$. Every nonempty set in $\mathcal{C}$ is generated by F, G, and H since $S_k = FG^kH(N)$.

THEOREM 3. *Let $\mathcal{C}$ be a denumerable field of sets which contains all finite sets. Suppose $A_0, \cdots, A_n$ is a partition of N into infinite sets of $\mathcal{C}$ none of which is the union of two disjoint infinite sets of $\mathcal{C}$. Then $\mathcal{C}$ is finitely generated if and only if $n > 0$.*

PROOF. If $n = 0$ then $\mathcal{C}$ is the field of all finite and cofinite sets. Suppose $\mathcal{C}$ were generated by $F_1, \cdots, F_k, G_1, \cdots, G_l$ where the range of each F is finite and the range of each G is cofinite. If G is obtained by composition from the G's then the range of G is cofinite. Hence if H is obtained by composition from the F's and G's then H has a finite range only if some F, say F_j, is used to obtain H. Then the number of elements in $H(N)$ is not more than the number of elements in $F_j(N)$. Hence the number of elements in the finite sets generated by $F_1, \cdots, F_k, G_1, \cdots, G_l$ is bounded which gives a contradiction. Thus for $n = 0$, $\mathcal{C}$ is not finitely generated.

Suppose $n > 0$. For $K \subseteq \{0, \cdots, n\}$, let $S_K = \bigcup_{k \in K} A_k$. Every set of $\mathcal{C}$ differs from some S_K by a finite number of elements. We will first show that every S_K can be generated by three functions A, B, and C which are compatible with $\mathcal{C}$. By Lemma 2, there is a function A such that $A(S) = A_0$ for every infinite set S in $\mathcal{C}$. A is compatible with $\mathcal{C}$ since $A(S)$ is either finite or A_0 for every S in $\mathcal{C}$.

Let B be a permutation which maps A_k onto A_{k+1} for $k = 0, \cdots, n-1$ and maps A_n onto A_0. B is compatible with $\mathcal{C}$ since B maps any set differing from S_K by a finite set onto a set differing from S_M by a finite set where M is obtained from K by adding 1 to each element of K other than n and replacing n by 0. Finally, let C be a function which is one-to-one on each A_i and maps A_i onto $A_0 \cup A_i$. Again C is compatible with $\mathcal{C}$ since C maps any set differing from S_K by a finite set onto a set differing from S_M by a finite set where $M = \emptyset$ if $K = \emptyset$ and $M = K \cup \{0\}$ if $K \neq \emptyset$.

Suppose t is the least element of a nonempty set $K \subseteq \{0, \cdots, n\}$. If K has just one element then $S_K = B^tA(N)$. If K has more than one element, let

$$K' = \{k : k + t \in K \wedge k > 0\}.$$

Then $S_K = B^tC(S_{K'})$. Hence by induction on the number of elements of K, for every nonempty set K, there is a function H obtained from B and C by composition such that $S_K = HA(N)$.

Now if S differs from S_K (K nonempty) by a finite set, then there is a set E which differs from A_0 by a finite set such that for some H

obtained from B and C by composition $H(\boldsymbol{E})=\boldsymbol{S}$. Indeed, we can choose H as above so that $H(\boldsymbol{A}_0)=\boldsymbol{S}_K$; since $H(\boldsymbol{N})=\boldsymbol{N}$ and $H^{-1}(j)$ is finite for all j, we can obtain $\boldsymbol{E}$ by modifying $\boldsymbol{A}_0$ by a finite set.

Thus, it remains to show that there are a finite number of functions compatible with $\mathcal{C}$ which generate every finite set and every set differing from $\boldsymbol{A}_0$ by a finite set.

Order the natural numbers in a sequence of the order type of the integers, say $\cdots, a_{-1}, a_0, a_1, \cdots$, so that $\boldsymbol{A}_0=\{a_0, a_{-1}, a_{-2}, \cdots\}$ and for $j>0$, $a_j\in \boldsymbol{A}_k$ if $j\equiv k \pmod{n}$ for $k=1, \cdots, n$. Now define three functions F, G, and H as follows:

$$
\begin{aligned}
F(a_k) &= a_{k+1}, \\
G(a_k) &= a_k \quad \text{if } k \leqq 0, \\
&= a_{k+1} \quad \text{if } k > 0, \\
H(a_k) &= a_0 \quad \text{if } k \leqq 0, \\
&= a_k \quad \text{if } k > 0.
\end{aligned}
$$

We will show that $\mathcal{C}$ is generated by A, B, C, F, F^{-1}, G and H. It is easy to check that F, F^{-1} G and H are all compatible with $\mathcal{C}$.

Now for $j, k, l, \cdots$ greater than 0,

$$F(\boldsymbol{A}_0\cup\{a_j, a_k, a_l, \cdots\}) = \boldsymbol{A}_0\cup\{a_1, a_{j+1}, a_{k+1}, a_{l+1}, \cdots\}$$

and

$$G(\boldsymbol{A}_0\cup\{a_j, a_k, a_l, \cdots\}) = \boldsymbol{A}_0\cup\{a_{j+1}, a_{k+1}, a_{l+1}, \cdots\}.$$

Hence by repeatedly applying F and G, we can obtain a function which maps $\boldsymbol{A}_0$ onto the union of an arbitrary finite set and $\boldsymbol{A}_0$. Next to obtain a set which differs from $\boldsymbol{A}_0$ by an arbitrary finite set, we apply a suitable power of F^{-1} to a set which is the union of a finite set and $\boldsymbol{A}_0$. On the other hand, if we first apply H to the union of $\boldsymbol{A}_0$ and a finite set and then apply a suitable power of F or F^{-1}, we can obtain an arbitrary nonempty finite set.

REMARK. This proof is due to R. M. Robinson. In the original proof, the number of generators depended on n.

LEMMA 4. *If $\boldsymbol{A}$ can be partitioned into arbitrarily many infinite sets of $\mathcal{C}$ and $\boldsymbol{A}=\boldsymbol{B}\cup\boldsymbol{C}$ where $\boldsymbol{B}$ and $\boldsymbol{C}$ are infinite sets of $\mathcal{C}$, then at least one of $\boldsymbol{B}$ and $\boldsymbol{C}$ can be partitioned into arbitrarily many infinite sets of $\mathcal{C}$.*

PROOF. Let $\boldsymbol{A}$ be partitioned into $2n$ infinite sets $\boldsymbol{S}_1, \cdots, \boldsymbol{S}_{2n}$ of $\mathcal{C}$. Then $\boldsymbol{B}=\bigcup(\boldsymbol{B}\cap\boldsymbol{S}_k)$ and $\boldsymbol{C}=\bigcup(\boldsymbol{C}\cap\boldsymbol{S}_k)$. Either $\boldsymbol{B}$ or $\boldsymbol{C}$ has at least n infinite parts under this partition of $\boldsymbol{A}$ and hence can be partitioned into n infinite sets of $\mathcal{C}$. Hence B or C can be partitioned into arbitrarily many infinite sets of $\mathcal{C}$.

THEOREM 4. *If $\mathcal{C}$ is a denumerable field of sets which contains all finite sets then either $\mathcal{C}$ is the field of all finite and cofinite sets or $\mathcal{C}$ is finitely generated.*

PROOF. Suppose there is a set A in $\mathcal{C}$ such that both A and $\bar{A}$ are infinite. If N can be partitioned into at most n infinite sets of $\mathcal{C}$ for some n, then Theorem 3 applies. Otherwise N can be partitioned into arbitrarily many infinite sets of $\mathcal{C}$ and by Lemma 4 so can A or $\bar{A}$. Now construct $A_0, A_1, \cdots$ in turn so that each A_j is an infinite set of $\mathcal{C}$, and $N - \bigcup_{j \leq k} A_j$ can always be partitioned into arbitrarily many infinite sets of $\mathcal{C}$. Of course, we must choose A_j to be disjoint from the earlier A's. Hence Theorem 2 applies and $\mathcal{C}$ is finitely generated.

REFERENCE

1. Julia Robinson, *Finite generation of recursively enumerable sets*, Proc. Amer. Math. Soc. **19** (1968), 1480–1486.

UNIVERSITY OF CALIFORNIA, BERKELEY