FINITELY GENERATED CLASSES OF SETS OF
NATURAL NUMBERS

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We say that a set $S$ of natural numbers is generated by a class $\mathcal{F}$ of
functions if $S$ is the range of a function $F$ obtained by composition
from the functions of $\mathcal{F}$. Here we consider the identity function $I$ to
be obtained by the empty composition. We consider only the case in
which $\mathcal{F}$ consists of functions of one variable on and to the set of
natural numbers $\mathbb{N}$. All sets considered are subsets of $\mathbb{N}$.

A class $\mathcal{C}$ of sets is generated by $\mathcal{F}$ if every nonempty set of $\mathcal{C}$ is
generated by $\mathcal{F}$ and every set generated by $\mathcal{F}$ is in $\mathcal{C}$. $\mathcal{C}$ is finitely
generated if there is a finite class of functions $\mathcal{F}$ which generates $\mathcal{C}$.

A function $F$ is compatible with $\mathcal{C}$ if $F(S)$ belongs to $\mathcal{C}$ for every
$S$ in $\mathcal{C}$.

Clearly, if $\mathcal{F}$ generates $\mathcal{C}$ then every $F$ in $\mathcal{F}$ is compatible with $\mathcal{C}$.
Also $\mathbb{N}$ belongs to $\mathcal{C}$. Furthermore to show that $\mathcal{F}$ generates $\mathcal{C}$ it is
sufficient to prove that every $F \in \mathcal{F}$ is compatible with $\mathcal{C}$, $\mathbb{N} \in \mathcal{C}$, and
every nonempty set in $\mathcal{C}$ is generated by $\mathcal{F}$.

In [1] finite sets of relatively simple functions are given which
generate the classes of recursively enumerable sets and diophantine
sets. Here we ask: What classes of sets can be finitely generated? In
particular, we will show that if $\mathcal{C}$ is a denumerable field of sets such
that all finite sets belong to $\mathcal{C}$ and at least one infinite set having an
infinite complement belongs to $\mathcal{C}$, then $\mathcal{C}$ is finitely generated.

**Lemma 1.** If $\mathcal{F}$ is an infinite class of infinite sets then there is a set $M$
such that $M \setminus T$ and $M \cap T$ are infinite for every $T$ belonging to $\mathcal{F}$.

**Proof.** Let each $T$ in $\mathcal{F}$ be listed infinitely often in the sequence
$T_0, T_1, \cdots$. We will construct $M$ and $\overline{M}$ simultaneously. At the $n$th
step put the least unassigned number of $T_n$ in $M$ and then the least
remaining unassigned number of $T_n$ in $\overline{M}$.

**Lemma 2.** Given infinite sets $T_0, T_1, \cdots$ and nonempty sets $S_0,
S_1, \cdots$ such that $S_0 = S_1$ whenever $T_0 \cap T_1$ is not empty. Then there is
a function $F$ such that $F(T_n) = S_n$.

**Proof.** Make a list of natural numbers such that each number
occurs infinitely often. We will use the list to define $F$ at one argument
for each step of the construction. Namely, if at a given step $n$ is the

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next number in the list and if \( x \) is the least unassigned number in \( T_n \) and \( y \) is the least number in \( S_n \) not yet in \( F(T_n) \), put \( Fx = y \); while if there is no such \( y \), put \( Fx \) the least number of \( S_n \). Finally, in case \( \bigcup T_k \neq N \), put \( Fx = 0 \) for every \( x \) not in any \( T_k \).

**Theorem 1. Hypotheses.** (1) \( C \) is a denumerable class of sets. (2) If \( A \) and \( B \) belong to \( C \) then \( A \cup B \) belongs to \( C \). (3) \( N \) belongs to \( C \). (4) Either (i) \( C \) contains all finite sets or (ii) \( C \) contains only infinite sets and \( C \) contains the set obtained from a member set by adjoining a single element. (5) There is an infinite partition \( \varnothing \subset C \) of \( N \) into infinite sets all of which are generated by a finite class \( \mathfrak{F} \) of functions compatible with \( C \).

**Conclusion.** \( C \) is finitely generated.

**Proof.** Let \( \varnothing = \{ P_0, P_1, \ldots \} \). We say that \( I \) is the index set of a set \( S \) if \( I = \{ n : S \cap P_n \neq \varnothing \} \). Let \( M \) be the set given by Lemma 1 such that every infinite index set of a set belonging to \( C \) contains infinitely many elements of both \( M \) and \( M \). (Concerning \( M \), we shall use only that \( M \) is infinite.) List the nonempty sets of \( C \) in a sequence \( T_0, T_1, \ldots \) such that \( T_m = N \) for every \( m \) in \( M \). We will define \( F \) in such a way that (a) \( F(T_n \cap P_k) = T_k \) whenever \( T_n \cap P_k \) is infinite, and (b) \( F(T_n) = N \) whenever \( T_n \) has an infinite index set. Let us check that this is sufficient. In the first place, every nonempty set of \( C \) will be generated by \( \{ F \} \cup \mathfrak{F} \) since \( F(P_k) = T_k \). Also \( F \) is compatible with \( C \), i.e. for every \( n \), \( F(T_n) \) belongs to \( C \). To see this, let \( I \) be the index set of \( T_n \). If \( I \) is infinite then \( F(T_n) = N \) and \( N \) belongs to \( C \) by (3). If \( I \) is finite, then

\[
F(T_n) = \bigcup_{k \in I} F(T_n \cap P_k),
\]

where \( F(T_n \cap P_k) \) is either finite or \( T_k \) according as \( T_n \cap P_k \) is finite or infinite. Hence if \( C \) contains all finite sets then \( F(T_n) \) is a nonempty finite union of sets of \( C \) and so \( F(T_n) \) belongs to \( C \). If \( C \) consists only of infinite sets, then \( T_n \cap P_k \) is infinite for some \( k \). Thus \( F(T_n) \) is the union of a nonempty finite union of sets of \( C \) and a finite set. Hence \( F(T_n) \) belongs to \( C \).

Therefore to complete the proof, we need only define \( F \) satisfying (a) and (b). Now make a list of pairs \( (n, k) \) so that each pair occurs infinitely often. We will use this list to index the steps in the definition of \( F \). At a step corresponding to \( (n, k) \), if there is a number in \( T_n \cap P_k \) at which \( F \) has not yet been assigned, let the least such number be \( x \). Put \( Fx \) equal to the least number in \( T_k \) which is not already in \( F(T_n \cap P_k) \) if there is such a number; otherwise put \( Fx \) equal to the
least number in $T_k$. Also if the index set of $T_k$ is infinite and $\gamma$ is the least number unassigned in $T_k \cap P_m$ for any $m$ in $M$, define $F_\gamma = k$. Then $F$ will satisfy both (a) and (b).

**Examples.** The classes of recursive sets, recursively enumerable sets, arithmetical sets, hyperarithmetical sets, are all finitely generated. We can let $P_k = (SD)^kD(N)$ since $S$, $D$ are compatible with each of these classes. Here $S$ is the successor function and $D$ the double function.

By the usual diagonal argument it is easy to see that the class of arithmetical sets cannot be generated by a finite class of arithmetical functions. (Similarly hyperarithmetical sets cannot be generated by a finite class of hyperarithmetical functions.) I do not know if the class of recursive sets can be generated by a finite number of recursive functions.

An interesting class of sets which is not covered by Theorem 1 is the class of polynomial sets and I do not know whether it is finitely generated. By a polynomial set, I mean the range of a polynomial with positive integer coefficients and any number of variables.

If we assume that $C$ is a field of sets which contains all finite sets, then condition (5) in Theorem 1 can be weakened to the mere existence of $\varphi$. In order to prove this, we shall need the following combinatorial result.

**Lemma 3.** Let $C$ be a denumerable field of sets which contains all finite sets. Suppose further that there is an infinite partition $\varphi_0$ of $N$ into infinite sets of $C$. Then there is an infinite partition $\varphi$ of $N$ into infinite sets $P_0, P_1, \cdots$ of $C$ such that for every $S$ in $C$ either $S$ or $\bar{S}$ is included in a finite union of the $P$'s.

**Remark.** This lemma is stronger than the one I originally proved and was suggested to me by Dana Scott. His proof was obtained by considering the Boolean algebra of sets of $C$ mod finite sets.

**Proof.** Let $S_0, S_1, \cdots$ be the sets of $C$. We will define $P_0, P_1, \cdots$ by means of a sequence of infinite partitions $\varphi_0, \varphi_1, \cdots$ of $N$ into infinite sets of $C$ such that for every $k$ and $n$ with $k < n$, $P_k$ belongs to $\varphi_n$ and either $S_k$ or $\bar{S_k}$ is included in $\bigcup_{j < n} P_j$. Assume that these conditions hold for all $n$ up through $t$. We shall define $P_t$ and $\varphi_{t+1}$ so that they also hold for $n = t+1$. Let $S$ be $S_t$ if $\bar{S}_t$ has infinitely many infinite parts under $\varphi_t$; otherwise let $S$ be $\bar{S}_t$. In either case, $\bar{S}$ has infinitely many infinite parts under $\varphi_t$. Let those infinite parts of $\bar{S}$ which are not contained in one of $P_0, \cdots, P_{t-1}$ be $I_0, I_1, \cdots$. Then put
Let $a_1, a_2, \ldots$ be the numbers which do not belong to

$$
\bigcup_{k \leq t} P_k \cup \bigcup_{j \neq 1} I_j,
$$

if there are any. Put $T_k = I_k \cup \{a_k\}$ if there is an $a_k$; otherwise let $T_k = I_k$. Finally, put

$$
\mathcal{P}_{t+1} = \{P_0, P_1, \cdots, P_t, T_1, \cdots\}.
$$

Clearly $P_t \subseteq \mathcal{P}_{t+1}$, $S \subseteq \bigcup_{k \leq t} P_k$, and $\mathcal{P}_{t+1}$ is an infinite partition of $\mathbb{N}$ into infinite sets of $\mathcal{C}$.

Now let $\mathcal{P} = \{P_0, P_1, \cdots\}$. By construction each $P_k$ is an infinite set of $\mathcal{C}$, $\mathcal{P}$ is disjoint, and for every $k$, either $S_k$ or $\overline{S}_k$ is contained in $\bigcup_{j \neq k} P_j$. Finally, every $n$ belongs to some $P_j$. In fact, $\{n\} = S_l$ for some $l$, so $n$ belongs to some $P_j$ with $j$ not exceeding $l$. Hence $\mathcal{P}$ satisfies the lemma.

**Theorem 2.** If a denumerable field of sets $\mathcal{C}$ contains all finite sets and includes an infinite disjoint class $\mathcal{P}$ of infinite sets $P_0, P_1, \cdots$, then $\mathcal{C}$ is finitely generated.

**Proof.** Without loss of generality, we may suppose that $\mathcal{P}$ is a partition of $\mathbb{N}$; otherwise adjoin the missing numbers to the parts $P_k$ with at most one to each $P_k$. By Lemma 3, we may also assume that $\mathcal{P}$ has the further property that for every $S$ in $\mathcal{C}$ either $S$ or $\overline{S}$ is included in a finite union of $P$'s.

By Lemma 2, we can define a function $\mathcal{H}$ so that $\mathcal{H}(S) = P_0$ for every infinite set $S$ in $\mathcal{C}$. Hence $\mathcal{H}$ is compatible with $\mathcal{C}$.

Let $G$ be defined so that $G(S) = P_{s+1}$ for every infinite set $S$ of $\mathcal{C}$ which is included in $P_n$. The existence of $G$ also follows from Lemma 2. Then $G^k(N) = P_k$ for all $k$. Furthermore, if $S$ is included in a finite union of $P$'s, then $G(S)$ is the union of a finite set and a finite union of $P$'s; if $\overline{S}$ is included in a finite union of $P$'s, then $G(S)$ is the union of a finite set and the complement of a finite union of $P$'s. In both cases $G(S)$ belongs to $\mathcal{C}$. Hence $G$ is compatible with $\mathcal{C}$.

We can now finish the proof by noticing the hypotheses of Theorem 1 hold. We can also construct $F$ more directly as follows: Let $S_0, S_1, \cdots$ be the nonempty sets of $\mathcal{C}$. By Lemma 2, there is a function $F$ such that $F(T) = S_k$ if $T$ is an infinite subset of $P_k$ which belongs to $\mathcal{C}$. Then $F$ is compatible with $\mathcal{C}$. Indeed, if $S$ is contained in a finite union of $P$'s then $F(S)$ is the union of a finite number of sets of $\mathcal{C}$ and
if \( S \) is contained in a finite union of \( P \)'s then \( F(S) = N \). Every non-empty set in \( \mathcal{C} \) is generated by \( F, G, \) and \( H \) since \( S_k = FG^kH(N) \).

**Theorem 3.** Let \( \mathcal{C} \) be a denumerable field of sets which contains all finite sets. Suppose \( A_0, \cdots, A_n \) is a partition of \( N \) into infinite sets of \( \mathcal{C} \) none of which is the union of two disjoint infinite sets of \( \mathcal{C} \). Then \( \mathcal{C} \) is finitely generated if and only if \( n > 0 \).

**Proof.** If \( n = 0 \) then \( \mathcal{C} \) is the field of all finite and cofinite sets. Suppose \( \mathcal{C} \) were generated by \( F_1, \cdots, F_k, G_1, \cdots, G_l \) where the range of each \( F \) is finite and the range of each \( G \) is cofinite. If \( G \) is obtained by composition from the \( G \)'s then the range of \( G \) is cofinite. Hence if \( H \) is obtained by composition from the \( F \)'s and \( G \)'s then \( H \) has a finite range only if some \( F \), say \( F_j \), is used to obtain \( H \). Then the number of elements in \( H(N) \) is not more than the number of elements in \( F_j(N) \). Hence the number of elements in the finite sets generated by \( F_1, \cdots, F_k, G_1, \cdots, G_l \) is bounded which gives a contradiction. Thus for \( n = 0 \), \( \mathcal{C} \) is not finitely generated.

Suppose \( n > 0 \). For \( K \subseteq \{0, \cdots, n\} \), let \( S_K = \bigcup_{k \in K} A_k \). Every set of \( \mathcal{C} \) differs from some \( S_K \) by a finite number of elements. We will first show that every \( S_K \) can be generated by three functions \( A, B, \) and \( C \) which are compatible with \( \mathcal{C} \). By Lemma 2, there is a function \( A \) such that \( A(S) = A_0 \) for every infinite set \( S \) in \( \mathcal{C} \). \( A \) is compatible with \( \mathcal{C} \) since \( A(S) \) is either finite or \( A_0 \) for every \( S \) in \( \mathcal{C} \).

Let \( B \) be a permutation which maps \( A_k \) onto \( A_{k+1} \) for \( k = 0, \cdots, n-1 \) and maps \( A_n \) onto \( A_0 \). \( B \) is compatible with \( \mathcal{C} \) since \( B \) maps any set differing from \( S_K \) by a finite set onto a set differing from \( S_M \) by a finite set where \( M \) is obtained from \( K \) by adding 1 to each element of \( K \) other than \( n \) and replacing \( n \) by 0. Finally, let \( C \) be a function which is one-to-one on each \( A_i \) and maps \( A_i \) onto \( A_0 \cup A_t \). Again \( C \) is compatible with \( \mathcal{C} \) since \( C \) maps any set differing from \( S_K \) by a finite set onto a set differing from \( S_M \) by a finite set where \( M = \emptyset \) if \( K = \emptyset \) and \( M = K \cup \{0\} \) if \( K \neq \emptyset \).

Suppose \( t \) is the least element of a nonempty set \( K \subseteq \{0, \cdots, n\} \). If \( K \) has just one element then \( S_K = B^tA(N) \). If \( K \) has more than one element, let

\[
K' = \{k: k + t \in K \wedge k > 0\}.
\]

Then \( S_K = B^tC(S_{K'}) \). Hence by induction on the number of elements of \( K \), \( S \) for every nonempty set \( K \), there is a function \( H \) obtained from \( B \) and \( C \) by composition such that \( S_K = HA(N) \).

Now if \( S \) differs from \( S_K \) (\( K \) nonempty) by a finite set, then there is a set \( E \) which differs from \( A_0 \) by a finite set such that for some \( H \)
obtained from $B$ and $C$ by composition $H(E) = S$. Indeed, we can choose $H$ as above so that $H(A_0) = S_k$; since $H(N) = N$ and $H^{-1}(j)$ is finite for all $j$, we can obtain $E$ by modifying $A_0$ by a finite set.

Thus, it remains to show that there are a finite number of functions compatible with $C$ which generate every finite set and every set differing from $A_0$ by a finite set.

Order the natural numbers in a sequence of the order type of the integers, say $\cdots, a_{-1}, a_0, a_1, \cdots$, so that $A_0 = \{a_0, a_{-1}, a_{-2}, \cdots \}$ and for $j > 0$, $a_j \in A_k$ if $j \equiv k \mod n$ for $k = 1, \cdots, n$. Now define three functions $F$, $G$, and $H$ as follows:

$$F(a_k) = a_{k+1},$$
$$G(a_k) = a_k \quad \text{if } k \leq 0,$$
$$= a_{k+1} \quad \text{if } k > 0,$$
$$H(a_k) = a_0 \quad \text{if } k \leq 0,$$
$$= a_k \quad \text{if } k > 0.$$  

We will show that $C$ is generated by $A$, $B$, $C$, $F$, $F^{-1}$, $G$, and $H$. It is easy to check that $F$, $F^{-1}$, $G$ and $H$ are all compatible with $C$.

Now for $j$, $k$, $l$, $\cdots$ greater than 0,

$$F(A_0 \cup \{a_j, a_k, a_l, \cdots \}) = A_0 \cup \{a_1, a_{j+1}, a_{k+1}, a_{l+1}, \cdots \}$$
and

$$G(A_0 \cup \{a_j, a_k, a_l, \cdots \}) = A_0 \cup \{a_{j+1}, a_{k+1}, a_{l+1}, \cdots \}.$$  

Hence by repeatedly applying $F$ and $G$, we can obtain a function which maps $A_0$ onto the union of an arbitrary finite set and $A_0$. Next to obtain a set which differs from $A_0$ by an arbitrary finite set, we apply a suitable power of $F^{-1}$ to a set which is the union of a finite set and $A_0$. On the other hand, if we first apply $H$ to the union of $A_0$ and a finite set and then apply a suitable power of $F$ or $F^{-1}$, we can obtain an arbitrary nonempty finite set.

**Remark.** This proof is due to R. M. Robinson. In the original proof, the number of generators depended on $n$.

**Lemma 4.** If $A$ can be partitioned into arbitrarily many infinite sets of $C$ and $A = B \cup C$ where $B$ and $C$ are infinite sets of $C$, then at least one of $B$ and $C$ can be partitioned into arbitrarily many infinite sets of $C$.

**Proof.** Let $A$ be partitioned into $2n$ infinite sets $S_1, \cdots, S_{2n}$ of $C$. Then $B = \bigcup (B \cap S_i)$ and $C = \bigcup (C \cap S_i)$. Either $B$ or $C$ has at least $n$ infinite parts under this partition of $A$ and hence can be partitioned.
into \( n \) infinite sets of \( \mathcal{C} \). Hence \( \mathcal{B} \) or \( \mathcal{C} \) can be partitioned into arbitrarily many infinite sets of \( \mathcal{C} \).

**Theorem 4.** If \( \mathcal{C} \) is a denumerable field of sets which contains all finite sets then either \( \mathcal{C} \) is the field of all finite and cofinite sets or \( \mathcal{C} \) is finitely generated.

**Proof.** Suppose there is a set \( A \) in \( \mathcal{C} \) such that both \( A \) and \( \overline{A} \) are infinite. If \( N \) can be partitioned into at most \( n \) infinite sets of \( \mathcal{C} \) for some \( n \), then Theorem 3 applies. Otherwise \( N \) can be partitioned into arbitrarily many infinite sets of \( \mathcal{C} \) and by Lemma 4 so can \( A \) or \( \overline{A} \).

Now construct \( A_0, A_1, \ldots \) in turn so that each \( A_j \) is an infinite set of \( \mathcal{C} \), and \( N - \bigcup_{j \geq k} A_j \) can always be partitioned into arbitrarily many infinite sets of \( \mathcal{C} \). Of course, we must choose \( A_j \) to be disjoint from the earlier \( A \)'s. Hence Theorem 2 applies and \( \mathcal{C} \) is finitely generated.

**Reference**