DETERMINANTAL RANK AND FLAT MODULES[1]

S. H. COX, JR.

1. By ring we mean commutative ring with identity. Module means unitary module. In this paper we use some results on determinantal rank to prove the following proposition: A finitely generated R-module M is projective if and only if M is flat and there is an exact sequence $0 \to M \to N \to L$ of R-modules such that N and L are projective (Theorem 2.9). A corollary is that a finitely generated R module M is projective if and only if M is flat, reflexive and $M^* = \mathrm{Hom}_R(M, R)$ is of finite presentation. In §3, we give an example of a cyclic ideal M in a ring R such that M is flat and reflexive, M^* is cyclic, but M is not projective.

We use f.g. in place of finitely generated and morphism instead of R-homomorphism. The set of prime ideals of a ring R is denoted Spec(R). $\boldsymbol{N}$ denotes the set of nonnegative integers. If $S \subset \boldsymbol{N}$ is unbounded, we write $\sup(S) = \infty$.

2. Let $u: M \to N$ be a morphism of R-modules. We define $\mathrm{rk}(u)$, the rank of u, by $\mathrm{rk}(u) = \sup\{n \in \boldsymbol{N}; \wedge^n u \neq 0\}$ where $\wedge^n$ denotes nth exterior power. We also define $\dim(M) = \mathrm{rk}(1_M)$. When M and N are f.g. free R-modules, $\mathrm{rk}(u)$ is also the determinantal rank of a matrix corresponding to u and $\dim(M)$ is the cardinality of a basis of M. When M and N are free we denote by $D(u, p)$ the ideal generated by the p-minors of a matrix corresponding to u. The ideals $\{D(u, p); p \in \boldsymbol{N}\}$ are the Fitting invariants of Coker(u) [**3**]. If S is a multiplicative system in R, then $\mathrm{rk}(u_S)$ is the rank of u_S as an R_S-morphism.

The following result from [2, p. 98, Exercise 3] will be used several times.

2.1. LEMMA. *Let M and N be f.g. free R-modules of dimensions m and n respectively. Then a morphism $u: M \to N$ is a monomorphism if and only if $m \leqq n$ and* $\mathrm{Ann}(D(u, m)) = 0$. (*In that case* $\mathrm{rk}(u) = m$.)

2.2. LEMMA. *Let $u: M \to N$ be a morphism of R-modules. Let S and T be multiplicative systems in R such that $S \leqq T$ (i.e. $R \to R_T$ factors through $R \to R_S$). Then*

(i) $\mathrm{rk}(u_T) \leqq \mathrm{rk}(u_S)$,

Received by the editors June 10, 1968.

[1] The contents of this paper form part of the author's 1968 Louisiana State University Ph.D. Dissertation. I wish to thank Professor R. L. Pendleton for his assistance in serving as my faculty advisor, and for his advice during the preparation of this paper.

(ii) *if M is f.g., then* $\exists f \in S \cdot \ni \cdot \mathrm{rk}(u_f) = \mathrm{rk}(u_S)$,
(iii) *if* $L \xrightarrow{v} M$ *is an R-morphism, then* $\mathrm{rk}(uv) \leqq \min\{\mathrm{rk}(u), \mathrm{rk}(v)\}$,
(iv) $\mathrm{rk}(u) = \sup\{\mathrm{rk}(u_p);\ p \in \mathrm{Spec}(R)\}$.

The proof of 2.2 is straightforward.

2.3. Proposition. *Let v: $M \to N$ and u: $N \to L$ be morphisms of f.g. free R-modules such that* $\mathrm{Image}(v) \supseteq \mathrm{Kernel}(u)$. *Then either* $\dim(N) \leqq \mathrm{rk}(u) + \dim(M)$ *or* $D(v, \dim(M)) \subseteq \sqrt{0}$.

Proof. Let $m = \dim(M)$, $n = \dim(N)$. Suppose that $D(v, m) \not\subseteq \sqrt{0}$. Since we wish to show that $n \leqq \mathrm{rk}(u) + m$, we may assume $m \leqq n$. As $D(v, m) \not\subseteq \sqrt{0}$, $\exists p \in \mathrm{Spec}(R) \cdot \ni \cdot D(v, m) \not\subseteq p$. Then it is easy to see that $D(v_p, m) = (D(v, m))_p = R_p$. Thus by [2, p. 98, Exercise 5] $(v_p)^*$: $(N_p)^* \to (M_p)^*$ is an epimorphism. Then by [1, p. 108, Proposition 6] v_p is a monomorphism onto a direct summand of N_p. Let $v_p(M_p) \oplus H = N_p$ and let c: $H \to N_p$ be the canonical inclusion. Evidently H is a free R_p-module of dimension $n - m$. Since $\mathrm{Image}(v_p) \supseteq \mathrm{Kernel}(u_p)$, $u_p c$ is a monomorphism. Hence $\mathrm{rk}(u_p c) = n - m$. By 2.2(i) and 2.2(iii), $\mathrm{rk}(u) \geqq n - m$.

2.4. Corollary. *If v in 2.3 is a monomorphism, then* $\dim(N) \leqq \mathrm{rk}(u) + \dim(M)$.

Proof. Let $D = D(v, \dim(M))$. By 2.1, $\mathrm{Ann}(D) = 0$. Thus, because D is f.g., $D \not\subseteq \sqrt{0}$. Hence, by 2.3, $\dim(N) \leqq \mathrm{rk}(u) + \dim(M)$.

2.5. Corollary. *Let u: $N \to L$ be a morphism of f.g. flat R-modules. Let M be a f.g. flat submodule of N such that* $M \supseteq \mathrm{Kernel}(u)$. *Then* $\dim(N) \leqq \mathrm{rk}(u) + \dim(M)$.

Proof. Let $p \in \mathrm{Spec}(R)$. By 2.4 and 2.2(i),

$$\dim(N_p) \leqq \mathrm{rk}(u_p) + \dim(M_p) \leqq \mathrm{rk}(u) + \dim(M).$$

Hence, by 2.2(iv), $\dim(N) \leqq \mathrm{rk}(u) + \dim(M)$.

2.6. Proposition. *Let v: $E \to F$, u: $F \to G$ be morphisms of f.g. free R-modules, $n = \dim(F)$. If $0 \leqq p$, $q \leqq n$ are integers such that $p + q > n$, and if $uv = 0$, then $D(v, q)D(u, p) = 0$.*

Proof. We may assume that $E = F = G$. Let $e_1, \cdots, e_n$ be a basis of E. Let $U = (u_{ij})$ and $V = (v_{ij})$ be the matrices of u and v, respectively, relative to $e_1, \cdots, e_n$. We use the notation of [2]. We must show that if $H, K, S, T \subseteq [1, n] \cdot \ni \cdot |H| = |K| = p$ and $|S| = |T| = q$, then $V_{S,T} U_{H,K} = 0$. Let H, K, S, $T \subseteq [1, n]$ with $|H| = |K| = p$ and $|S| = |T| = q$. We will construct an endomorphism w of $E \cdot \ni \cdot \det(w)$

$=tV_{S,T}U_{H,K}$ for some non-zero-divisor t of R. Then we will show $\det(w)=0$. For $L\subseteq[1, n]$, $\pi_L\colon E\to E$ is the projection defined by $\pi_L(e_i)=e_i$ if $i\in L$ and $\pi_L(e_i)=0$ if $i\in L'$. Select one-to-one correspondences $\sigma\colon K'\to H'$ and $\tau\colon T'\to S'$. Let $\alpha=\sigma^{-1}$ and $\beta=\tau^{-1}$. Define $f_\sigma\colon E\to E$ by $f_\sigma(e_i)=e_{\sigma(i)}$ if $i\in K'$, and $f_\sigma(e_i)=0$ if $i\in K$. Define f_τ, f_α and f_β similarly. It is easy to check that

(i) $\pi_S+\pi_{S'}=1_E=\pi_K+\pi_{K'}$,

(ii) $f_\tau f_\beta=\pi_{S'}, f_\alpha f_\sigma=\pi_{K'}$,

(iii) $\pi_T f_\beta=0=f_\sigma\pi_H=\pi_H f_\sigma=\pi_S f_\tau$,

(iv) $f_\beta v\pi_T+1_E$ and $\pi_H u f_\alpha+1_E$ are monomorphisms.

Now let $w=(\pi_H u+f_\sigma)(v\pi_T+f_\tau)$. Using (i)–(iii), we get that $\pi_H u+f_\sigma=(\pi_H u f_\alpha+1_E)(\pi_H u\pi_K+f_\sigma)$. By 2.1, $t_1=\det(\pi_H u f_\alpha+1_E)$ is a non-zero-divisor of R. By Laplace's Expansion, $\det(\pi_H u\pi_K+f_\sigma)=\pm U_{H,K}$. Hence, $\det(\pi_H u+f_\sigma)=\pm t_1 U_{H,K}$. Similarly, $\det(v\pi_T+f_\tau)=\pm t_2 V_{S,T}$ where $t_2=\det(f_\beta v\pi_T+1_E)$ is a non-zero-divisor of R. Hence, $\det(w)=tV_{S,T}U_{H,K}$ where t is a non-zero-divisor of R. Now let W be the matrix of w relative to $e_1, \cdots, e_n$. We have

$$\det(W) = \rho_{T,T'}\Sigma_L\, \rho_{L,L'}W_{L,T}W_{L',T'}$$

by Laplace's Expansion. Let $L\subseteq[1,n]\cdot\ni\cdot |L|=|T|=q$. Then $L\cap H\neq\varnothing$ since $|H|=p$ and $p+q>n$. Choose $j\in L\cap H$. Then $\pi_{\{j\}}w\pi_T=0$. Hence, the jth row of the $q\times q$ submatrix of W determined by rows in L and columns in T is zero. Therefore $W_{L,T}=0$, and $\det(W)=0$. Finally, $U_{H,K}V_{S,T}=0$.

2.7. COROLLARY. *If* $0\to M\xrightarrow{v}F\xrightarrow{u}G$ *is an exact sequence of f.g. free R-modules, then* $\dim(M)+\mathrm{rk}(u)=\dim(F)$.

PROOF. By 2.4, $\dim(F)\leqq\dim(M)+\mathrm{rk}(u)$. By 2.6, $D(u, n-m+1)\cdot D(v, m)=0$ where $n=\dim(F)$ and $m=\dim(M)$. By 2.1, $\mathrm{Ann}(D(v, m))=0$. So $D(u, n-m+1)=0$. Therefore, $\wedge^{n-m+1}u=0$, i.e., $\mathrm{rk}(u)\leqq n-m$.

2.8. PROPOSITION. *If* $M\xrightarrow{v}F\xrightarrow{u}G$ *is an exact sequence of f.g. free R-modules such that* $\mathrm{rk}(u)+\dim(M)=\dim(F)$ *and if* $\mathrm{Image}(v)$ *is flat, then v is a monomorphism.*

PROOF. Let $p\in\mathrm{Spec}(R)$. Since $v(M)$ is flat, $v_p(M_p)$ is a free R_p-module. By 2.7, $\dim(v_p(M_p))+\mathrm{rk}(u_p)=\dim(F_p)$. By 2.2(i), $\mathrm{rk}(u_p)\leqq\mathrm{rk}(u)$. Thus,

$$\begin{aligned}\dim(M_p) \geqq \dim(v_p(M_p)) &= \dim(F_p) - \mathrm{rk}(u_p) \geqq \dim(F_p) - \mathrm{rk}(u)\\ &= \dim(F) - \mathrm{rk}(u) = \dim(M) = \dim(M_p).\end{aligned}$$

So $M_p\to v_p(M_p)$ is an epimorphism of free R_p-modules of the same dimension. Hence, $M_p \to v_p(M_p)$ is an isomorphism. Therefore v_p is a monomorphism, $\forall p \in \mathrm{Spec}(R)$. Therefore v is a monomorphism.

2.9. THEOREM. *A finitely generated R-module M is projective if and only if M is flat and there is an exact sequence* $0 \to M \xrightarrow{v} F \xrightarrow{u} G$ *with F and G projective R-modules.*

PROOF. If M is projective, then certainly M is flat and such a sequence exists. Conversely, it is not hard to see that we may assume that F and G are free and f.g. Now let $p \in \mathrm{Spec}(R)$. By 2.2.(ii), $\exists f' \in R \backslash p \cdot \ni \cdot \mathrm{rk}(u_{f'}) = \mathrm{rk}(u_p)$. Let $m = \dim(M_p)$. By 2.7, $m + \mathrm{rk}(u_p) = \dim(F_p)$. Let E be a free R-module of dimension m. There is an R-morphism $E \xrightarrow{w} M \cdot \ni \cdot w_p$ is an epimorphism (in fact an isomorphism). By [2, p. 136, Proposition 2], $\exists f'' \in R \backslash p \cdot \ni \cdot w_{f''}$ is an epimorphism. Let $f = f'f''$. Then $\mathrm{rk}(u_f) = \mathrm{rk}(u_p)$, w_f is an epimorphism, and $f \in R \backslash p$. The sequence $E_f \to F_f \xrightarrow{u} G_f$ is an exact sequence of f.g. free R_f-modules; $\mathrm{Image}(w_f) = M_f$ is a flat R_f-module;

$$\mathrm{rk}(u_f) + \dim(E_f) = \mathrm{rk}(u_p) + \dim(M_p) = \dim(F_p) = \dim(F_f).$$

Hence, by 2.8, w_f is a monomorphism. Therefore, $M_f = \mathrm{Image}(w_f)$ is a free R_f-module. We have shown that $\forall p \in \mathrm{Spec}(R) \exists f \in R \backslash p \cdot \ni \cdot M_f$ is a free R_f-module. Thus M is projective by [1, p. 138, Theorem 1].

2.10. COROLLARY. *A finitely generated R-module M is projective if and only if M is flat, reflexive and* M^* *is of finite presentation.*

PROOF. The necessity is well known. For the converse, let $f: M \to M^{**}$ be the canonical morphism.

Since M^* is of finite presentation, there is an exact sequence $E \xrightarrow{g} F \xrightarrow{h} M^* \to 0$ with E and F f.g. free R-modules. Hence, $0 \to M \xrightarrow{h^*f} F^* \xrightarrow{g^*} E^*$ is exact and F^* and E^* are free. By 2.9, M is projective.

3. Let S be a ring which admits a commutative S-algebra $A \neq 0$ satisfying

(1) there is a non-zero-divisor t of S such that $tA = 0$,

(2) $\forall a \in A \exists b \in A \cdot \ni \cdot ba = a$,

(3) A has no multiplicative identity.

Let $R = S \times A$ with the usual coordinate addition and multiplication defined by $(s, a)(r, b) = (sr, ra + sb + ab)$. Fix a non-zero-divisor t of S such that $tA = 0$. Let $r = (t, 0)$ and $M = Rr$. Denote the exact sequence $0 \to \mathrm{Ann}_R(M) \to R \to M \to 0$ by (E). M is flat: it is sufficient to show $x \in \mathrm{Ann}_R(M) \Rightarrow \exists y \in \mathrm{Ann}_R(M) \cdot \ni \cdot yx = x$ [1, p. 65, Exercise 23]. Let $x \in \mathrm{Ann}_R(M)$. Write $x = (s, a)$. Since $xr = 0$ and t is a non-zero-divisor, $s = 0$. By (2), $\exists b \in A \cdot \ni \cdot ba = a$. Let $y = (0, b)$. Since $tA = 0$, $yr = 0$ so

$y \in \mathrm{Ann}_R(M)$. Also $yx=(0, b)(0, a)=(0, ba)=(0, a)=x$. Thus M is flat. M is not projective: if M is projective, then (E) splits. Hence $\mathrm{Ann}_R(M)$ is generated by an idempotent e of R. Since $A\neq 0$, $e\neq 0$. Write $e=(0, u)$. Let $c\in A$. Then $(0, c)r=0$ so $(0, c)\in \mathrm{Ann}_R(M)$. Therefore, $(0, c)=e(0, c)=(0, u)(0, c)=(0, uc)$. That is $uc=c$, $\forall c\in A$ contradicting (3). Thus M is a cyclic flat nonprojective ideal of R.

Now consider the following choice for S and A. S is the ring of integers. I is an infinite set and A is the set of functions $f\colon I\to S/(2)$ such that $f(i)=0$ for all but finitely many $i\in I$. With pointwise operations, A is an S-algebra satisfying (1)–(3) with $t=2$ in (1). Thus $M=R(2, 0)$ is flat but not projective. It is easy to see that in this case we have $M=\mathrm{Ann}_R(\mathrm{Ann}_R(M))$. It follows that M^* is cyclic and M is reflexive. Hence, M is a cyclic flat reflexive nonprojective ideal with M^* cyclic. This shows that the hypothesis "M^* is of finite presentation" in 2.10 cannot be replaced by "M^* is f.g."

References

1. N. Bourbaki, *Éléments de mathématique. Algèbre commutative*, Chapters 1 and 2, Hermann, Paris, 1961.

2. ———, *Éléments de mathématique. Algèbre*, Chapter 3, Hermann, Paris, 1958.

3. H. Fitting, *Die Determinantenideale eines Moduls*, Jber. Deutsch. Math.-Verein. 46 (1936), 195–228.

Louisiana State University