ON SUBGROUPS OF FINITE SOLVABLE GROUPS

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In this note, the word “group” means “a finite solvable group.” Let $G$ be a group, and $D$ a system normalizer of $G$. In [5] we introduced the subgroup $Q(D)$, generated by all subgroups of $G$ in which $D$ is subnormal. In this note we use one of the alternative characterizations of $Q(D)$, as given in [5], to define an analogue, $Q(H)$, for arbitrary subgroups $H$ of $G$. We derive a covering-avoidance characterization of $Q(H)$, and deduce that it is homomorphism invariant. These results, in turn, can be used to shorten many of the proofs in [5].

We first recall some definitions. A Sylow system $\mathcal{S}$ of $G$ is said to reduce into $H$, if $\mathcal{S}\cap H$ (i.e. the set of intersections of members of $\mathcal{S}$ with $H$) is a Sylow system of $H$. An $H$-composition-series of $G$ is a series

\[ \{1\} = G_n \triangle G_{n-1} \triangle \cdots \triangle G_1 \triangle G_0 = G \]

in which each $G_i$ is a maximal $H$-invariant normal subgroup of $G_{i-1}$. The groups $G_i/G_{i+1}$ are referred to as $H$-composition-factors of $G$. If $H$ induces (by conjugation) only the trivial automorphism on $G_i/G_{i+1}$, then the latter is $H$-central, otherwise it is $H$-eccentric. The product of the indices $|G_i:G_{i+1}|$, for those factors in (1) which are $H$-central and are avoided by $H$, is denoted by $z_0(H)$. Here a subgroup $K$ covers $G_i/G_{i+1}$ if $G_i \subseteq G_{i+1}K$, $K$ avoids $G_i/G_{i+1}$ if $G_i \cap K \subseteq G_{i+1}$. $z_0(H)$ is an invariant of $H$ (and $G$), i.e. it does not depend on the series (1) (see [2]).

Let $M$ be a set of Sylow systems of $G$. We refer to $M$ as a block, if $M$ is disjoint from all of its conjugates (so that if we consider $G$ as a permutation group on its Sylow systems, the conjugates of $M$ form an imprimitivity system).

Now let $H$ be any subgroup of $G$. We denote by $M_0$ the smallest block of $G$ which contains all the Sylow systems reducing into $H$.

**Definition.** The stabilizer of $M_0$ (i.e. the set of all $g \in G$ such that $M_0^g = M_0$) is denoted by $Q(H)$.

**Theorem 1.** $Q(H)$ covers all $H$-central $H$-composition-factors of $G$. Moreover, if $K \supseteq H$ and $K$ covers all $H$-central $H$-composition-factors, then $K \supseteq Q(H)$.

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Proof. Let $G_i/G_{i+1}$ be an $H$-central factor in (1), and let $\mathcal{S}$ be a Sylow system of $G$ reducing into $H$. Then $\mathcal{S}$ reduces into $G_iH$ [3, Lemma 2.7]. Let $D$ be $\mathcal{N}_{G_iH}(\mathcal{S}\cap G_iH)$. Then $D$ transforms $\mathcal{S}$ into systems reducing into $H$ (because they all have the same intersection with $G_iH$), and thus $D$ stabilizes $\mathcal{M}_0$, and $D \subseteq Q(H)$. Since $D$ covers the central factor $G_i/G_{i+1}$ of $G_iH$, $Q(H)$ covers $G_i/G_{i+1}$.

Now let $K \supseteq H$, and assume that $K$ covers all $H$-central factors. A $K$-central factor is certainly $H$-central, so $K$ covers all of its central factors, and thus $K$ is abnormal (see [2, §2]; an abnormal subgroup is one for which $g \in \langle K, K^g \rangle$ for all $g \in G$). The intersections of $K$ with the terms of (1) form an $H$-composition series of $K$, and as $K$ covers all $H$-central factors in (1), these give rise to $H$-central factors of $K$ of the same order. Thus $z_0(H)$, computed in $K$, is the same as $z_0(H)$, computed in $G$.

Let $D$ be a system normalizer of $G$, and $D_i$ one of $K$. By [2, p. 541] there are $|H|/|D| \cdot z_0(H)$ Sylow systems of $G$ reducing into $H$, $|H|/|D_i| \cdot z_0(H)$ systems of $K$ reducing into $H$, and each system of $K$ is the intersection with $K$ of $|D_i|/|D|$ systems of $G$. It follows that the number of systems of $G$ reducing into both $K$ and $H$ is

$$\frac{|D_i|}{|D|} \cdot \frac{|H|}{|D_i|} \cdot z_0(H) = \frac{|H|}{|D|} \cdot z_0(H)$$

i.e. all systems of $G$ reducing into $H$ reduce also into $K$. Let $\mathcal{M}$ be the set of all Sylow systems reducible into $K$. Then, $K$ being abnormal, $\mathcal{M}$ is a block with stabilizer $K$ [5, Lemma 2]. Thus $\mathcal{M} \supseteq \mathcal{M}_0$, and the stabilizer of $\mathcal{M}_0$ is contained in the stabilizer of $\mathcal{M}$.

Remark 1. It is seen from the proof that it is enough to assume that $K$ covers the $H$-central factors in a given series (1).

Remark 2. For each central factor $G_i/G_{i+1}$ in (1), let $D_i$ be a system normalizer of $G_iH$, as in the first paragraph of the proof. Then we have seen that $D_i \subseteq Q(H)$, and $D_i$ covers $G_i/G_{i+1}$. Thus Theorem 1 implies that $Q(H) = \langle H, D_i \rangle$ (i ranges over all indices such that $G_i/G_{i+1}$ is $H$-central).

Remark 3. Take $K = Q(H)$ in the above proof. Then $\mathcal{M} \supseteq \mathcal{M}_0$. If $\mathcal{S} \subseteq \mathcal{M}_0$ and $g \in Q(H)$, then $\mathcal{S}^g \subseteq \mathcal{M}_0$. Take $\mathcal{S}$ to reduce into $H$, then we have seen that $\mathcal{S}$ reduces into $Q(H)$, and all systems reducing into $Q(H)$ are conjugate under $Q(H)$ by [1, Lemma, p. 360]; thus $\mathcal{M} \subseteq \mathcal{M}_0$ and $\mathcal{M}_0$ is the set of all Sylow systems reducing into $Q(H)$.

Theorem 2. Let $G \to G^*$ be an epimorphism, and let stars denote epimorphic images. Then $Q(H^*) = Q(H)^*$.

Proof. Let $N$ be the kernel of the epimorphism, and let $R/N = Q(H)^*$, $Q = Q(H)$. We may assume that $N$ is one of the terms in
(1). Then $Q^*$ covers all $H^*$-central factors in the $H^*$-composition-series $\{G_i^*\}$ of $G^*$. Thus $Q^* \supseteq R$. In turn, $R$ covers all $H$-central factors in (1), so $R \supseteq Q$, and $R^* = Q^*$.

Suppose $H \triangle \triangle L$, and let $\mathfrak{N}$ be the set of Sylow systems reducible into $L$. Then all systems in $\mathfrak{N}$ reduce into $H$, so $\mathfrak{N} \subseteq \mathfrak{M}_0$. As $L$ stabilizes $\mathfrak{N}$, $L \subseteq Q(H)$. In general, $Q(H)$ is not generated by all such $L$, as we can see by taking $H$ to be any self-normalizing subgroup that is not abnormal.

Now take $D$ to be any subgroup normalizing the Sylow system $\mathfrak{S}$ of $G$. In the notations of Remark 2, $D \subseteq D_i$ for each of the $i$'s considered there. Thus $Q(D) = \langle D_i \rangle$, and $D \triangle D_i$, as each $D_i$ is nilpotent. So $Q(D)$ is generated by all subgroups in which $D$ is subnormal. If $D \subseteq E$ and $E$ is nilpotent, then $D \triangle E$, hence $E \subseteq Q(D)$. On the other hand, the subgroups $D_i$ are nilpotent. We thus see that $Q(D)$ is, indeed, the subgroup introduced in [5], and at the same time we have alternative proofs for the properties of $Q(D)$ discussed there (the present treatment is slightly more general, as we allow $D$ to be an arbitrary subgroup of a system normalizer).

As a further application, consider the problem: when is $\mathfrak{M}_0$ the set of all systems reducing into $H$? Suppose this is the case. By Remark 3, all systems of $Q(H)$ reduce into $H$, so that $H \triangle \triangle Q(H)$ [2] or [4]. We then have in $Q(H)$, and therefore also in $G$, $e_0(H) = \left| Q(H) : H \right|$. Thus $Q(H)$ is the strong subnormalizer of $H$, in the sense of [5]. Conversely, assume that $H \triangle \triangle L$ and that $\left| L : H \right| = e_0(H)$. $L$ covers or avoids all factors in (1), and the ones that $L$ covers but $H$ avoids must be $H$-central (they are $H$-isomorphic to factors between $L$ and $H$). By orders, $L$ covers all $H$-central factors, so $L \supseteq Q(H)$, $L = Q(H)$, and $L$ is necessarily the strong subnormalizer of $H$. Since $H \triangle \triangle L$, all systems of $L$ reduce into $H$, so $\mathfrak{M}_0$ is indeed the desired set of Sylow systems. We have thus reproved Theorems 3 and 4 of [5], while Theorem 5 there follows from our present Theorem 2.

References


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