ON THE UNIFORMIZATION OF SOUSLIN $\mathfrak{F}$ SETS

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1. Introduction. A set E in a cross product space $X \times Y$ is said to be uniformized by a set U if $U \subset E$; the projections $\Pi_X E$ and $\Pi_X U$ of E and U onto X coincide; and for each $x \in \Pi_X E$ the set $(\{x\} \times Y) \cap U$ of points of U above x is a singleton. (For a detailed discussion of the problem of uniformization see C. A. Rogers and R. C. Willmott [6].) M. Kondô [2] showed that if E is the complement of a Souslin $\mathfrak{F}$ set in the Euclidean plane (or, more generally, the cross product of two complete separable metric spaces), then a uniformizing set U may be chosen which is again the complement of a Souslin $\mathfrak{F}$ set. He showed further [2, Theorem 7, p. 230] that if E is a Souslin $\mathfrak{F}$ set then U may be chosen to be a set from the projective class $B_2 = P_2 \cap C_2$, where P_2 is the family of projections of complements of Souslin $\mathfrak{F}$ sets and C_2 is the family of complements of sets of P_2.

In a recent paper, C. A. Rogers and R. C. Willmott [6], studying the first result of Kondô, removed almost all conditions on the space X. They also considered the case where Y is a descriptive Borel set (see C.A. Rogers [4]), obtaining a partial uniformization result (Theorem 19) in terms of a complement of a Souslin $\mathfrak{F}$ set. M. Sion [8], studying the case of an analytic set (which is a Souslin $\mathfrak{F}$ set in a complete separable metric space; see, e.g. Bressler and Sion [1, Theorem 5.7]) also removed almost all conditions on X and some on Y, and concentrated his attention on the properties of the uniformizing set as a function, e.g. measurable for a large class of measures, not concerning himself with its set-theoretic properties.

In this paper, using methods of Rogers and Willmott [6], the second result of Kondô is generalized in Theorem 2 by removing all conditions on X, and by finding for a Souslin $\mathfrak{F}$ set E a uniformizing set U which is the difference between E and a $\mathfrak{D}_\sigma$-set, where $\mathfrak{D}$ is the family of differences of Souslin $\mathfrak{F}$ sets. U is a member of the family of $\mathfrak{D}_{\sigma\delta}$-sets, a subset of B_2 in a complete separable metric space. It is known that there exist Souslin $\mathfrak{F}$ sets in the plane which cannot be uniformized either by a Souslin $\mathfrak{F}$ set or by the complement of a Souslin $\mathfrak{F}$ set (see, e.g. [7, p. 57]). The case where Y is a descriptive Borel set is also considered and a result, Theorem 3, is obtained for a Souslin $\mathfrak{F}$ set using the difference between the Souslin $\mathfrak{F}$ set and a

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$\mathfrak{D}_\sigma$-set, which is analogous to that obtained in [6, Theorem 19], for the complement of a Souslin $\mathfrak{F}$ set using the complement of a Souslin $\mathfrak{F}$ set.

I am grateful to Professor C. A. Rogers for an observation which led to a reduction of the type of the uniformizing set in these theorems.

2. **Definitions and notation.** All definitions and notation are those of [6]. The following should suffice for those parts of this paper which are independent or are statements of theorems.

I denotes the family of sequences of positive integers. For $i \in I$, $i|n$ is the n-tuple $i_1, i_2, \cdots, i_n$; and

$$I_{i|n} = \{j \in I: j|n = i|n\},$$

called a Baire interval of order n. The Baire intervals form a base for a topology on I; with this topology I is a homeomorph of the irrational numbers.

Souslin $\mathfrak{F}$ in X is the family of sets A having a representation of the form

$$A = \bigcup_{i \in I} \bigcap_{n=1}^{\infty} A(i|n),$$

where each set $A(i|n)$ is closed in X. This Souslin representation is said to be disjoint if for each $i, j \in I$,

$$\bigcap_{n=1}^{\infty} A(i|n) \cap \bigcap_{n=1}^{\infty} A(j|n) = \emptyset \qquad \text{if } i \neq j.$$

In a complete separable metric space, Souslin $\mathfrak{F}$ coincides with the classical family of analytic sets (see, e.g. Bressler and Sion [1, Theorem 5.7]). Souslin sets with disjoint representations are studied in [6, §§4, 5 and 8].

Borel Souslin $\mathfrak{F}$ in X is the smallest family containing Souslin $\mathfrak{F}$ in X and closed under complementation and countable unions.

A function K from I to the compact subsets of a topological space X is said to be semicontinuous if for each $i_0 \in I$ and open set G of X with $K(i_0) \subset G$, there exists an open set 0 of I with $I_0 \in 0$ and $K[0] \subset G$. A set in a Hausdorff space X is called descriptive Borel if it has a representation of the form $K[I] = \bigcup_{i \in I} K(i)$ where K is a semicontinuous function from I to the compact subsets of X such that $K(i) \cap K(j) = \emptyset$ if $i \neq j$. For properties of descriptive Borel sets see C. A. Rogers [4].

3. **Theorems.** We establish our main result in $X \times I$ and subse-

quently use mapping techniques of Rogers and Willmott [6] to extend it to other spaces.

Let $\mathfrak{D}$ be the family of differences of Souslin $\mathfrak{F}$ sets. Then the uniformizing set in the following theorem

$$U = A \sim \bigcup_{n=1}^{\infty} W_n = \bigcap_{n=1}^{\infty} (A \sim W_n)$$

is an $\mathcal{E}_\delta$-set, where $\mathcal{E}$ is the family of sets E of the form

$$E = A \sim (B \sim C) = (A \sim B) \cup (A \cap B \cap C),$$

where A, B and C are Souslin $\mathfrak{F}$ sets. Clearly

$$\mathcal{E}_\delta \subset \mathfrak{D}_{\sigma\delta} \subset \text{Borel Souslin } \mathfrak{F}.$$

In a complete separable metric space, B_2 contains Souslin $\mathfrak{F}$, is closed under complementation and countable unions (see, e.g. [7, p. 21]) and hence contains Borel Souslin $\mathfrak{F}$.

THEOREM 1. *Let X be a topological space. Then a Souslin $\mathfrak{F}$ set A in $X \times I$ can be uniformized by the difference between A and a $\mathfrak{D}_\sigma$-set.*

PROOF. Let A be a Souslin $\mathfrak{F}$ set in $X \times I$. Then, by the first part of the proof of Theorem 17 in [6], A has a representation

$$A = \bigcup_{i \in I} \bigcap_{n=1}^{\infty} F(i \mid n),$$

where for each $i \in I$ and $n \geqq 1$,

$$F(i \mid n) = B(i \mid n) \times C(i \mid n),$$

where $B(i \mid n)$ is closed in X and $C(i \mid n)$ is empty or is a Baire interval in I of order $\geqq n$. We may suppose that for each $i \in I$ and $n \geqq 1$, $C(i \mid n+1) \subset C(i \mid n)$, for otherwise, setting

$$F'(i \mid n) = B(i \mid n) \times \bigcap_{m=1}^{n} C(i \mid m),$$

we have the $F'(i \mid n)$ of the required form since two Baire intervals are either disjoint or one is contained in the other, and clearly

$$\bigcup_{i \in I} \bigcap_{n=1}^{\infty} F'(i \mid n) = \bigcup_{i \in I} \bigcap_{n=1}^{\infty} F(i \mid n) = A.$$

Now for $i \in I$, $m \geqq 1$, set

$$A(i \mid m) = \bigcup_{j \in I_{i \mid m}} \bigcap_{n=1}^{\infty} F(j \mid n) \subset A \cap F(i \mid m).$$

Let

$$V_1 = \bigcup_{n_1=2}^{\infty}\left[A(n_1)\cap\left(\bigcup_{p=1}^{n_1-1}\Pi_X A(p)\times I\right)\sim\bigcup_{p=1}^{n_1-1}A(p)\right],$$

$$V_2 = \bigcup_{n_1=1}^{\infty}\bigcup_{n_2=2}^{\infty}\left[A(n_1, n_2)\cap\left(\bigcup_{p=1}^{n_2-1}\Pi_X A(n_1, p)\times I\right)\right.$$
$$\left.\sim\left(\bigcup_{p=1}^{n_2-1}A(n_1, p)\cup\bigcup_{p=1}^{n_1-1}A(p)\right)\right],$$

and, in general,

$$V_s = \bigcup_{n_1=1}^{\infty}\cdots\bigcup_{n_{s-1}=1}^{\infty}\bigcup_{n_s=2}^{\infty}$$
$$\cdot\left[A(n_1, \cdots, n_s)\cap\left(\bigcup_{p=1}^{n_s-1}\Pi_X A(n_1, \cdots, n_{s-1}, p)\times I\right)\right.$$
$$\sim\left(\bigcup_{p=1}^{n_s-1}A(n_1, \cdots, n_{s-1}, p)\cup\bigcup_{p=1}^{n_{s-1}-1}A(n_1, \cdots, n_{s-2}, p)\right.$$
$$\left.\left.\cdots\cup\bigcup_{p=1}^{n_1-1}A(p)\right)\right].$$

Finally, set $U=A\sim\bigcup_{s=1}^{\infty}V_s$ and check:

(1) As $I_{i|m}$ is homeomorphic to I, each $A(i|m)$ is a Souslin $\mathfrak{F}$ set in $X\times I$, and so, by the result in [5], $\Pi_X A(i|m)$ is Souslin $\mathfrak{F}$ in X, as is $\Pi_X A(i|m)\times I$ in $X\times I$. Hence each V_s is a $\mathfrak{D}_\sigma$-set, as is $\bigcup_{s=1}^{\infty}V_s$.

(2) $U\subset A$ trivially.

(3) $\Pi_X A=\Pi_X U$. By (2) it suffices to show that $\Pi_X A\subset\Pi_X U$. Suppose $x\in\Pi_X A$. As $A=\bigcup_{p=1}^{\infty}A(p)$, there exists a least integer, i_1, such that $x\in\Pi_X A(i_1)$.

By construction of V_1 and the choice of i_1, V_1 contains all points of A outside $A(i_1)$ which project onto x, but contains no points of $A(i_1)$ which project onto x. That is

$$\Pi_X^{-1}(x)\cap(A\sim V_1) = \Pi_X^{-1}(x)\cap A(i_1)\neq\varnothing.$$

As $A(i_1)\subset F(i_1)=B(i_1)\times C(i_1)$, we have

$$\Pi_X^{-1}(x)\cap A(i_1)\subset\{x\}\times C(i_1),$$

and $x\in B(i_1)$, $C(i_1)\neq\varnothing$.

Again, $A(i_1)=\bigcup_{p=1}^{\infty}A(i_1, p)$, and there exists a first integer, i_2, such that $x\in\Pi_X A(i_1, i_2)$. By construction of V_2 and the choice of i_1 and i_2, V_2 contains all points of $A(i_1)$ outside $A(i_1, i_2)$ which project

onto x, but contains no points of $A(i_1, i_2)$ which project onto x. Then $V_1 \cup V_2$ contains all points of A outside $A(i_1, i_2)$ which project onto x, but contains no points of $A(i_1, i_2)$ which project onto x, i.e.

$$\Pi_X^{-1}(x) \cap [A \sim (V_1 \cup V_2)] = \Pi_X^{-1}(x) \cap A(i_1, i_2) \neq \emptyset.$$

Again $\Pi_X^{-1}(x) \cap A(i_1, i_2) \subset \{x\} \times C(i_1, i_2)$, $x \in B(i_1, i_2)$, and $C(i_1, i_2) \neq \emptyset$. Continuing, we obtain $i \in I$ such that for each $n \geqq 1$,

$$\Pi_X^{-1}(x) \cap \left[A \sim \left(\bigcup_{s=1}^{n} V_s\right)\right] = \Pi_X^{-1}(x) \cap A(i \mid n) \neq \emptyset,$$

$$\Pi_X^{-1}(x) \cap A(i \mid n) \subset \{x\} \times C(i \mid n),$$

$x \in B(i \mid n)$, and $C(i \mid n) \neq \emptyset$.

Now, as $C(i \mid n+1) \subset C(i \mid n)$, and as $C(i \mid n)$ is a Baire interval of order $\geqq n$, it follows that $\bigcap_{n=1}^{\infty} C(i \mid n)$ is a singleton. Let $\bigcap_{n=1}^{\infty} C(i \mid n) = \{y\}$. Then

$$(x, y) \in \bigcap_{n=1}^{\infty} [B(i \mid n) \times C(i \mid n)] = \bigcap_{n=1}^{\infty} F(i \mid n) = \bigcap_{n=1}^{\infty} A(i \mid n).$$

Then for all n,

$$(x, y) \in \Pi_X^{-1}(x) \cap A(i \mid n) = \Pi_X^{-1}(x) \cap \left[A \sim \bigcup_{s=1}^{n} V_s\right]$$

and so

$$(x, y) \in \bigcap_{n=1}^{\infty} \left(A \sim \bigcup_{s=1}^{n} V_s\right) = A \sim \bigcup_{s=1}^{\infty} V_s = U.$$

Hence $x \in \Pi_X U$ and $\Pi_X A \subset \Pi_X U$.

(4) Finally, for $x \in \Pi_X A = \Pi_X U$, we have from a formula in (3), for each n,

$$\Pi_X^{-1}(x) \cap \left[A \sim \bigcup_{s=1}^{n} V_s\right] \subset \{x\} \times C(i \mid n),$$

and hence

$$\Pi_X^{-1}(x) \cap U = \bigcap_{n=1}^{\infty} \left[\Pi_X^{-1}(x) \cap \left(A \sim \bigcup_{s=1}^{n} V_n\right)\right]$$

$$\subset \bigcap_{n=1}^{\infty} [\{x\} \times C(i \mid n)] = \{x\} \times \bigcap_{n=1}^{\infty} C(i \mid n),$$

so that $\Pi_X^{-1}(x) \cap U$ is a singleton. This completes the proof.

The next theorem generalizes Kondô's result on the uniformization of a Souslin $\mathfrak{F}$ set in the product of two complete separable metric spaces. It is well known that a complete separable metric space is a continuous one-to-one image of a closed subset of $\boldsymbol{I}$ (see, e.g. [3, p. 443]).

THEOREM 2. *Suppose Y is a continuous one-to-one image of a closed subset $\boldsymbol{H}$ of $\boldsymbol{I}$, X is a topological space, and E is a Souslin $\mathfrak{F}$ set in $X\times Y$. Then E can be uniformized by the difference between E and a $\mathfrak{D}_\sigma$-set.*

PROOF. Let g be a continuous function defined on $\boldsymbol{H}$ which maps $\boldsymbol{H}$ one-to-one onto Y. Define $\phi: X\times\boldsymbol{H}\to X\times Y$ by setting

$$\phi(x, i) = (x, g(i)) \qquad \text{for all } x\in X, \quad i\in \boldsymbol{H}.$$

Then, as in the proof of Theorem 18 in [6], ϕ is a continuous one-to-one function on $X\times\boldsymbol{H}$ onto $X\times Y$ which preserves Souslin $\mathfrak{F}$ sets, i.e. if B is a Souslin $\mathfrak{F}$ set in $X\times\boldsymbol{H}$, then $\phi[B]$ is Souslin $\mathfrak{F}$ in $X\times Y$. As ϕ is continuous, $A=\phi^{-1}[E]$ is a Souslin $\mathfrak{F}$ set in $X\times\boldsymbol{H}$ and hence in $X\times\boldsymbol{I}$ as $\boldsymbol{H}$ is closed in $\boldsymbol{I}$. By Theorem 1, there is a $\mathfrak{D}_\sigma$-set, V, in $X\times\boldsymbol{I}$ such that $A\sim V$ uniformizes A. We have $V\subset A\subset X\times\boldsymbol{H}$ and evidently V is also a $\mathfrak{D}_\sigma$-set in $X\times\boldsymbol{H}$. Clearly

$$U = \phi[A\sim V] = \phi[A]\sim\phi[V] = E\sim\phi[V]$$

uniformizes E. Finally, $\phi[V]$ is a $\mathfrak{D}_\sigma$-set in $X\times Y$ since ϕ preserves Souslin $\mathfrak{F}$ sets and unions, and being one-to-one, also preserves differences.

THEOREM 3. *Let X be a topological space, Y a Hausdorff space with a representation $Y=K[\boldsymbol{I}]$ as a descriptive Borel set. Suppose that each open set in $X\times Y$ has a disjoint Souslin representation. Let A be a Souslin $\mathfrak{F}$ set in $X\times Y$. Then there is a set U which is the difference between A and a $\mathfrak{D}_\sigma$-set in $X\times Y$, and which satisfies*

(a) $U\subset A$,

(b) $\Pi_X A=\Pi_X U$, *and*

(c) *for each $x\in\Pi_X A$, the set $\Pi_Y[\Pi_X^{-1}(x)\cap U]$ is compact and contained in some set $K(i)$ with $i\in\boldsymbol{I}$.*

PROOF. Let A have the representation

$$A = A[\boldsymbol{I}], \qquad A(i) = \bigcap_{n=1}^{\infty} A(i\,|\,n),$$

the sets $A(i\,|\,n)$ being closed. Then, as in the proof of Theorem 19 in [6], let

$$X \times Y = F[I], \qquad F(i) = \bigcap_{n=1}^{\infty} F(i \mid n)$$

be a representation of $X \times Y$ such that the sets $F(i \mid n)$ are closed, and each set $A(i \mid n)$ is the union of those fragments $F(i)$ that it meets, and such that for each $i \in I$, there is a $j \in I$ such that for each integer $m \geqq 1$, there exists an integer n with

$$F[I_{i|n}] \subset X \times K[I_{j|m}].$$

As in the proof of that theorem we define a map ω: $X \times Y \to X \times I$ by setting for each $x \in X$, $y \in Y$, $\omega(x, y) = (x, i)$, where $i = i(x, y)$ is the unique i in I with $(x, y) \in F(i)$. Then again ωA is a Souslin $\mathfrak{F}$ set in $X \times I$. By Theorem 1, there exist Souslin $\mathfrak{F}$ sets B_s, C_s in $X \times I$ such that

$$W = \omega A \sim \bigcup_{s=1}^{\infty} (B_s \sim C_s)$$

uniformizes ωA. Let $U = \omega^{-1}[W]$. Properties (a), (b), and (c) follow for U as in the proof of Theorem 19 in [6]. It remains to show that U is the difference between A and a $\mathfrak{D}_\tau$-set in $X \times Y$. We have

$$\begin{aligned} U &= \omega^{-1}\left[\omega A \sim \bigcup_{s=1}^{\infty} (B_s \sim C_s)\right] \\ &= \omega^{-1}[\omega A] \sim \left(\bigcup_{s=1}^{\infty} (\omega^{-1}[B_s] \sim \omega^{-1}[C_s])\right). \end{aligned}$$

Using Theorem 8 of [6] as in the proof of Theorem 19 in [6], we have $\omega^{-1}[B_s]$ and $\omega^{-1}[C_s]$ Souslin $\mathfrak{F}$ sets in $X \times Y$. Finally, by definition of ω, $\omega^{-1}[\omega F(i)] = F(i)$ for i in I, and as each set $A(i \mid n)$ is the union of the fragments $F(i)$ that it meets the same is true of each $A(i) = \bigcap_{n=1}^{\infty} A(i \mid n)$, and so also $\omega^{-1}[\omega A(i)] = A(i)$ for i in I. Hence

$$\omega^{-1}[\omega A] = \omega^{-1}\left[\omega \bigcup_{i \in I} A(i)\right] = \bigcup_{i \in I} \omega^{-1}[\omega A(i)] = \bigcup_{i \in I} A(i) = A.$$

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