ON THE ZEROS OF THE RIEMANN ZETA-FUNCTION

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1. Introduction. If

\[ R(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s), \]

the functional equation for the Riemann zeta-function \( \zeta(s) \) is given by \( R(s) = R(1-s) \). Berlowitz [1] has recently shown that if \( 0 < \lambda < 1 \), then both \( \text{Re} \ R(\lambda + it) \) and \( \text{Im} \ R(\lambda + it) \) vanish infinitely often. In this note we shall give an improvement on this result.

Let \( N_R(\lambda, T) \) denote the number of zeros for \( \text{Re} \ R(\lambda + it) \) on \( 0 < t < T \). Similarly, \( N_I(\lambda, T) \) denotes the number of zeros for \( \text{Im} \ R(\lambda + it) \) on \( 0 < t < T \).

**Theorem.** If \( 0 < \lambda < 1 \), then

\[ (1.1) \quad N_R(\lambda, T) > AT \]

and

\[ (1.2) \quad N_I(\lambda, T) > AT, \]

where \( A = A(\lambda) \).

Here and elsewhere \( A, A_1 \) and \( A_2 \) denote positive constants and \( K_1 \) and \( K_2 \) complex constants, none of which is necessarily the same with each occurrence.

For \( \lambda = \frac{1}{2} \), the result follows from a famous theorem of Hardy and Littlewood [2, p. 222]. In fact, we use their method [2, pp. 222–226] in our proof.

2. Proof of the theorem. We prove (1.1) for the case \( 0 < \lambda < \frac{1}{2} \). The proof for \( \frac{1}{2} < \lambda < 1 \) will then follow from the functional equation \( R(s) = R(1-s) \).

From [1] we have

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re} \ R(\lambda + it) e^{it} dt = (e^{-\lambda t} + e^{-(1-\lambda) t}) \psi(e^{-\pi t}) - \frac{1}{2} (e^{\lambda t} + e^{(1-\lambda) t}), \]

where \( \psi(x) = \sum_{n=1}^{\infty} \exp(-n^2 \pi x) \) and \( \text{Re} \ e^{-\pi t} > 0 \). Putting

\[ \xi = -i(\pi/4 - \delta/2) - \gamma, \quad \delta > 0, \]

we see that

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\[ F(t) = \frac{1}{(2\pi)^{1/2}} \text{Re} R(\lambda + it)e^{(\pi/4-\delta/2)t} \]

and

\[ f(y) = (e^{(\pi/4-\delta/2)+y} + e^{(1-\lambda)/(\pi/4-\delta/2)+y})\psi(e^{(\pi/2-\delta)+y}) \]

\[ - \frac{1}{2}(e^{-\lambda(i/4-\delta/2)+y} + e^{-(1-\lambda)/(i/4-\delta/2)+y}) \]

are Fourier transforms. We now use formula (10.7.1), [2, p. 223], i.e.

\[ \int_{-\infty}^{\infty} \left| \int_{t}^{t+H} F(u) du \right|^2 dt \leq 2H^2 \int_{0}^{1/H} |f(y)|^2 dy \]

\[ + 8 \int_{1/H}^{\infty} |f(y)|^2 y^{-2} dy, \]

where \( H \geq 1 \) is a constant to be chosen later. Letting \( y = \log x \) and \( G = e^{1/H} \), we see that (2.1) yields

\[ \int_{-\infty}^{\infty} \left| \int_{t}^{t+H} F(u) du \right|^2 dt \]

\[ = O \left\{ H^2 \int_{1}^{G} |\psi(e^{(\pi/2-\delta)x^2})|^2 dx \right\} \]

\[ + O \left\{ \int_{G}^{\infty} |\psi(e^{(\pi/2-\delta)x^2})|^2 \frac{x^{1-2\lambda}}{\log^2 x} dx \right\} + O(H). \]

The first integral on the right side of (2.2) is estimated in [2, p. 224], and is \( O(H\delta^{-1/2}) \) as \( \delta \) tends to 0.

To estimate the second we write

\[ |\psi(e^{(\pi/2-\delta)x^2})|^2 \]

\[ = \sum_{n=1}^{\infty} \exp[-2n^2\pi x^2 \sin \delta] \]

\[ + \sum_{m=1}^{\infty} \sum_{n=1 \atop mn \neq 0} \exp[-(m^2 + n^2)\pi x^2 \sin \delta + i(m^2 - n^2)\pi x^2 \cos \delta]. \]

To examine the contribution of the first sum on the right side of (2.3), divide the interval \((G, \infty)\) into the subintervals \((G, \delta^{-1/2})\) and \((\delta^{-1/2}, \infty)\) and let the corresponding integrals be denoted by \( I_1 \) and \( I_2 \), respectively. The sum under consideration is \( O(x^{-1}\delta^{-1/2}) \), and so
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\[ I_1 = O \left( \delta^{-1/2} \int_{\sigma-1/2}^{\sigma+1/2} \frac{x^{1-2\lambda}}{x \log^2 x} \, dx \right) \]
\[ = O(H\delta^{-1}), \]

upon an integration by parts. The sum is also \( O(\exp(-2\pi x^2\delta)) \) for \( x^2\delta \geq 1 \), and so

\[ I_2 = O \left( \int_{\sigma-1/2}^{\sigma+1/2} \frac{x^{1-2\lambda}}{\log^2 x} \exp(-2\pi x^2\delta) \, dx \right) \]
\[ = O \left( \delta \int_{\sigma-1/2}^{\sigma+1/2} x \exp(-2\pi x^2\delta) \, dx \right) \]
\[ = O(\delta^{-1}). \]

The contribution of the second term on the right side of (2.3) is \( O(H\delta^{-1/2}) \) by the same argument as that in [2, p. 224].

Letting \( \delta = 1/T \) and

\[ I = \int_t^{t+H} \text{Re} R(\lambda + iu) e^{(\sigma/4-1/T)u} \, du, \]

we have shown that

\[ (2.4) \quad \int_T^{2T} |I|^2 \, dt = O(HT^{1-\lambda}). \]

From the functional equation for \( \xi(s) \),

\[ \text{Re} R(\lambda + it) = \frac{1}{2} \left\{ R(\lambda + it) + R(\lambda - it) \right\} \]
\[ = \frac{1}{2} \left\{ R(\lambda + it) + R(1 - \lambda + it) \right\}. \]

Using the above and Stirling’s formula for \( \Gamma(\sigma + it) \), we find that

\[ | \text{Re} R(\lambda + it) | \geq e^{-\pi t/4} \left| K_1 t^{\lambda - 1/2} \xi(\lambda + it) + K_2 t^{\lambda - 3/2} (1 - \lambda + it) \right|. \]

Thus, if \( T \leq t \leq 2T \),

\[ J = \int_t^{t+H} | \text{Re} R(\lambda + iu) | e^{(\sigma/4-1/T)u} \, du \]
\[ \geq T^{\lambda - 1/2} \int_t^{t+H} \left| K_1 \xi(\lambda + iu) + K_2 u^{1/2 - \lambda} \xi(1 - \lambda + iu) \right| \, du \]
\[ \geq T^{\lambda - 1/2} \left| \int_t^{t+H} \{ K_1 \xi(\lambda + iu) + K_2 u^{1/2 - \lambda} \xi(1 - \lambda + iu) \} \, du \right|. \]
Using a simple approximation for $\xi(s)$ [2, p. 67], we have

$$T^{(1-\lambda)/2} J \geq \left| \int_t^{t+H} \left\{ K_1 \left( \sum_{n \leq AT} n^{-\lambda-\xi u} + O(T^{-\lambda}) \right) \\
+ K_2 u^{1/2-\lambda} \left( \sum_{n \leq AT} n^{\lambda-1-\xi u} + O(T^{-\lambda}) \right) \right\} \, du \right|$$

$$\geq (A_1 + A_2 T^{1/2-\lambda}) H + O \left\{ \left| \int_t^{t+H} \sum_{2 \leq n \leq AT} n^{-\lambda-\xi u} du \right| \right\}$$

$$+ O \left\{ \left| \int_t^{t+H} u^{1/2-\lambda} \sum_{2 \leq n \leq AT} n^{\lambda-1-\xi u} du \right| \right\} + O(HT^{-\lambda}).$$

Employing the second mean value theorem for integrals to the second integral on the right side, we have

$$T^{(1-\lambda)/2} J \geq A_1 T^{1/2-\lambda} H + O(HT^{-\lambda})$$

$$+ O \left\{ \left| \sum_{2 \leq n \leq AT} (1/n^{\lambda+\xi(t+H)} \log n - 1/n^{\lambda+\xi t} \log n) \right| \right\}$$

$$+ O \left\{ T^{1/2-\lambda} \left| \sum_{2 \leq n \leq AT} (1/n^{1-\lambda+\xi(t+H)} \log n - 1/n^{1-\lambda+\xi t} \log n) \right| \right\}$$

$$\geq A_1 T^{1/2-\lambda} H + \Psi,$$

say, where $t < r < t + H$.

We show next that

$$\int_T^{2T} |\Psi|^2 \, dt = O(T^{2-2\lambda}).$$

Clearly, it is sufficient to examine

$$\int_T^{2T} \left| \sum_{2 \leq n \leq AT} 1/n^{\lambda+\xi t} \log n \right|^2 \, dt$$

$$= \int_T^{2T} \sum_{2 \leq m \leq AT} 1/m^{\lambda+\xi t} \log m \sum_{2 \leq n \leq AT} 1/n^{\lambda-\xi t} \log n \, dt$$

$$= T \sum_{2 \leq n \leq AT} 1/n^{2\lambda} \log^2 n + \sum_{2 \leq m \leq AT} \sum_{2 \leq n \leq AT} (1/(mn)^{\lambda} \log m \log n) \int_T^{2T} (n/m)^{t+\xi t} \, dt$$

$$= O(T^{2-2\lambda}) + O \left( \sum \sum_{2 \leq m \leq AT} 1/(mn)^{\lambda} \log m \log n \log n/m \right).$$

This last sum is estimated in exactly the same manner as (7.2.1),
[2, p. 116], and is $O(T^{2-\beta})$. Replacing $\lambda$ by $1-\lambda$ in the above calculation, we find

$$\int_{T}^{2T} \left| \sum_{2 \leq n \leq AT} \frac{1}{n^{1-\lambda+i\tau}} \right|^2 dt = O(T^{\rho}).$$

Thus, we have proved (2.6).

Now let $S$ denote the subset of $(T, 2T)$ where $I = J$. Thus,

$$\int_{S} | I | dt = \int_{S} J dt.$$

From (2.4),

$$\int_{S} | I | dt \leq \int_{T}^{2T} | I | dt \leq \left\{ T \int_{T}^{2T} | I |^2 dt \right\}^{1/2} \leq A_{4} T^{1/2} T^{1-\lambda/2}.$$

Also, from (2.5) and (2.6),

$$\int_{S} J dt \geq T^{(\rho-1)/2} \int_{S} (A_{1} T^{1/2-\lambda} H + \Psi) dt$$

$$\geq A_{1} H T^{-\lambda/2} m(S) - T^{(\rho-1)/2} \int_{T}^{2T} | \Psi | dt$$

$$\geq A_{1} H T^{-\lambda/2} m(S) - T^{(\rho-1)/2} \left\{ T \int_{T}^{2T} | \Psi |^2 dt \right\}^{1/2}$$

$$\geq A_{1} H T^{-\lambda/2} m(S) - AT^{1-\lambda/2},$$

where $m(S)$ denotes the measure of $S$. Combining the above with (2.7), we find that

$$A_{1} H T^{-\lambda/2} m(S) \leq AT^{1-\lambda/2} + A_{2} H^{1/2} T^{1-\lambda/2},$$

or

$$m(S) \leq A H^{-1/2} T.$$

Divide $(T, 2T)$ into $\left[ T/2H \right]$ pairs of abutting intervals $j_{1}, j_{2},$ each, except for possibly the last $j_{2},$ of length $H.$ Then, if $j_{1}$ does not contain entirely points of $S$, either $j_{1}$ or $j_{2}$ contains a zero of $\text{Re} R(\lambda+i\tau).$ Thus, if there are $\nu j_{2}$-intervals containing only points of $S$, $\nu H \leq m(S)$, and $\text{Re} R(\lambda+i\tau)$ has at least

$$\left[ T/2H \right] - \nu > T/3H - AT/H^{3/2} > T/4H$$

zeros if $H$ is large enough.
To prove (1.2) we merely observe that
\[ i \operatorname{Im} R(\lambda + it) = \frac{i}{2} \left\{ R(\lambda + it) - R(1 - \lambda + it) \right\}, \]
and then proceed as before.

REFERENCES