Let $F$ be a field of algebraic functions of one variable having the finite field $K$ as exact field of constants. The class number of $F$ is defined as the order of the finite group, $C_0(F)$, of divisor classes of degree zero. Let $L$ be the unique cyclic extension of $K$ of degree $n$, $E = F \cdot L$ the corresponding constant extension with galois group $G$. Since $K$ is perfect, the canonical homomorphism of the group of divisor classes of $F$ in the group of divisor classes of $E$ is an injection [2, p. 477]. If $h_E$, $h_F$ denote the class numbers of $E$ and $F$, respectively, we have $h_E = h_F \cdot k$, for some integer $k$. The purpose of this note is to prove the following two theorems:

**Theorem 1.** If $E/F$ is a constant extension of the algebraic function field $F$ and if $G$ is the corresponding galois group, then $H^i(G, C_0(E)) = 0$ for all $i$.

**Theorem 2.** If $E/F$ is a constant extension of prime degree $p$, then $h_E = h_F \cdot k$ where $k \equiv 1 \mod p$ if $p \mid h_F$ and $k \equiv 0 \mod p$ if $p \nmid h_F$ and $t$ is the $p$-rank of $C_0(F)$.

Throughout this note $E/F$ will denote a cyclic constant extension of the algebraic function field $F$ with galois group $G$ generated by $\sigma$. In a natural fashion $G$ operates on the prime divisors of $E$. Thus if $N_{E/F} = 1+\sigma + \cdots + \sigma^{n-1}$, $n = [E:F]$, we have for a prime $\wp$ of $E$, that $N_{E/F}\wp = \wp^{n(\wp)}$ where $\wp = \wp_1\wp_2\cdots\wp_t$, $n(\wp) = [E(\wp):F(\wp)]$ the degree of the corresponding completions since the extension is everywhere unramified. We see easily then that $\deg_{E/K} N_{E/F}(\wp) = n \deg_{B/E}\wp$. This norm map extends from the prime divisors to the full divisor group $D(E)$ in the natural way and it is compatible with the field norm and formation of principal divisors: that is, if we use parentheses to denote principal divisors and $\alpha \in E$, then $N_{E/F}(\alpha) = (N_{E/F}\alpha)$. We shall write the group operation in $D(E)$ additively and denote as usual the subgroup of principal divisors by $P(E)$ and the divisors of degree zero by $D_0(E)$. Similar notations will be used for the field $F$.

**I. Proof of Theorem 1.** In the normal extension $E/F$ all primes $\wp$ of $E$ extending a fixed prime $p$ of $F$ are conjugate. Thus for $a \in D_0(E)$ we have $a^\sigma = a$ for all $\sigma \in G$ if and only if $a \in D_0(F)$; therefore $D_0(E)^\sigma$
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\[ \text{From the exact } G\text{-sequence } 0 \rightarrow L \rightarrow E \rightarrow P(E) \rightarrow 0 \text{ we get the exact sequence} \]
\[ 0 \rightarrow L^0 \rightarrow E^0 \rightarrow P(E)^G \rightarrow H^1(G, L) \rightarrow H^1(G, E) \rightarrow \cdots \text{. Using Hilbert's Theorem 90 and the fact that } L \text{ is a finite field, we conclude} \]
\[ P(E)^G = P(F) \quad \text{and} \quad H^1(G, P(E)) = 0. \]

Consequently from the exact G-sequence \[ 0 \rightarrow P(E) \rightarrow D_0(E) \rightarrow C_0(E) \rightarrow 0 \text{ and (1) we derive} \]
\[ C_0(E)^G = C_0(F). \]

We next claim that the induced map \[ N_{E,F}: C_0(E) \rightarrow C_0(F) \] is surjective. Let \( J(E) \) denote the idèle group of \( E \) and define \( \phi: J(E) \rightarrow D(E) \) by \( \phi(A) = \sum v_\varphi(A) A_\varphi \) where \( v_\varphi(A) = v_\varphi(A_\varphi), \) \( A_\varphi \) the \( \varphi \) component of the idèle \( A. \phi(A) \) is a divisor since \( A \) is a unit almost everywhere. It is easily checked that \( \phi \) is surjective. Let \( J(E)^0 = \phi^{-1}(D_0(E)) \).

Recall that \( J(E) \) is also a \( G \)-module and the norm on idèles is compatible with the norm on divisors \[ [4] \]. Thus we have the exact and commutative diagram:
\[ J(E)^0 \rightarrow D_0(E) \rightarrow 0 \]
\[ N_{E,F} \downarrow \quad \downarrow N_{E,F} \]
\[ J(F)^0 \rightarrow D_0(F) \rightarrow 0 \]

If \( I(E) \) denotes the idèle class group of \( E \), then class field theory \[ [1, \text{p. 79}] \] asserts that \( J(F)^0/F \subset N_{E,F}(I(E)). \) Now suppose \( a \in C_0(F) \) and \( a \in D_0(F) \) is a representative for \( a \). Let \( A \in J(F)^0 \) be such that \( \phi(A) = a. \) Then there exists \( B \in J(E)^0 \) such that \( N_{E,F}B = A(\beta), \beta \in F. \) If \( b \) is the class of \( \phi(B) \) in \( C_0(E) \), we conclude from (3) that \( N_{E,F}b = a. \) Hence \( N_{E,F} \) is surjective.

We have therefore proved \( H^0(G, C_0(E)) = 0. \) But since \( C_0(E) \) is a finite group the Herbrand quotient gives that \( H^1(G, C_0(E)) = 0 \) as well. Using that \( G \) is cyclic and consequently has periodic cohomology, we have proved Theorem 1.

II. **Proof of Theorem 2.** Since \( C_0(E)^G = C_0(F) \) we can induce a \( G \)-action on the factor group \( C_0(E)/C_0(F) \) and get the exact \( G \)-sequence
\[ 0 \rightarrow C_0(F) \rightarrow C_0(E) \rightarrow C_0(E)/C_0(F) \rightarrow 0. \]

From this we derive
\[ 0 \rightarrow C_0(F)^G \rightarrow C_0(E)^G \rightarrow (C_0(E)/C_0(F))^G \rightarrow H^1(G, C_0(F)) \rightarrow \cdots. \]
Using Theorem 1 and the trivial action of $G$ on $C_0(F)$ we see that

\[(4) \quad (C_0(E)/C_0(F))^G \cong H^1(G, C_0(F)).\]

Furthermore in the case of trivial action we have [3, p. 142]

\[(5) \quad |H^1(G, C_0(F))| = p^t, \quad \text{where } t \text{ is the } p\text{-rank of } C_0(F).\]

Therefore if $p \nmid h_F$, we have immediately that for $k = \left|\frac{C_0(E)}{C_0(F)}\right|$, $k \equiv 0 \mod p^t$, since $(C_0(E)/C_0(F))^G$ is a subgroup of $C_0(E)/C_0(F)$.

On the other hand, taking the decomposition of $C_0(E)/C_0(F)$ into $G$ orbits we see that $k = \sum [G: H_c]$ where the summation extends over a set of representatives for the various orbits and $H_c$ is the corresponding stabilizer. Therefore if $G$ is of prime order $p$, we have $[G: H_c] = 1$ or $p$ and $[G: H_c] = 1$ if and only if $G = H_c$. Tracing the action of $G$ on $C_0(E)/C_0(F)$, we see that this is the case if and only if $c \in (C_0(E)/C_0(F))^G$. Therefore we have

\[(6) \quad k = \left|(C_0(E)/C_0(F))^G\right| + sp.\]

Hence if $p \mid h_F$ from (4), (5) and (6) we conclude that $k \equiv 1 \mod p$.

The following remarks are immediate consequences of Theorem 2.

1. If $F$ is a function field in one variable with finite field of constants $k$ and $p$ is a prime with $p^\alpha || h_F$, $\alpha \geq 1$, then there is a constant extension $E/F$ with $p^{\alpha+1} \mid h_E$.

2. If $E/L$ is a constant extension of the algebraic function field $F/K$ of prime degree $p$ then $p \mid h_E$ if and only if $p \mid h_F$ ($h_F \neq 1$).

**Bibliography**