WEIGHT SPACES AND IRREDUCIBLE REPRESENTATIONS OF SIMPLE LIE ALGEBRAS

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Let L denote a simple Lie algebra over an algebraically closed field K of characteristic zero. Harish-Chandra [2] has shown that to each one-dimensional representation (λ, K) of the Cartan subalgebra $\mathcal{H}$ of L we may associate an irreducible representation of L admitting λ as a "highest weight function". This has been generalized by the author [3] by showing that if $\mathcal{C}$ is the centralizer of $\mathcal{H}$ in U, the universal enveloping algebra of L, and (γ, K) is a one-dimensional representation of $\mathcal{C}$ we may again construct an irreducible representation of L admitting $\gamma \downarrow \mathcal{H}$ as a weight function. In this paper we make a further study of the relationship between the representations of L and their weight spaces.

1. **Weight spaces of irreducible representations.** It is well known that there exists a one-one correspondence between the representations of L and those of U which preserves irreducibility. Throughout this paper we will not distinguish between a representation of L and its unique extension to U.

DEFINITION 1. Let (ρ, V) be a representation of L. Then for each linear functional $\lambda \in \mathcal{H}^*$, the dual linear space of the Cartan subalgebra, we define

$$V_\lambda = \{v \in V \mid \rho(H)v = \lambda(H)v \ (\forall H \in \mathcal{H})\}.$$

Also we denote by $[\rho:\lambda]$ the dimension of the subspace V_λ. If, in particular, $[\rho:\lambda] > 0$ we call λ a *weight function* of the representation ρ and V_λ the corresponding *weight space.*

It is clear from the definitions of $\mathcal{C}$ and V_λ that for any element $c \in \mathcal{C}$, $\rho(c)$ maps V_λ into V_λ.

DEFINITION 2. Let (ρ, V) be a representation of L and $\lambda \in \mathcal{H}^*$. Then we define a representation $(\eta(\rho, \lambda), V_\lambda)$ of $\mathcal{C}$ by setting $\eta(\rho, \lambda)(c) = \rho(c) \downarrow V_\lambda$ for all $c \in \mathcal{C}$.

LEMMA 1. *If (ρ, V) is an irreducible representation of L and $\lambda \in \mathcal{H}^*$ such that $[\rho:\lambda] > 0$ then $(\eta(\rho, \lambda), V_\lambda)$ is a nontrivial, irreducible representation of $\mathcal{C}$.*

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PROOF. It suffices to show that for an arbitrary nonzero element $v \in V_\lambda$ we have $\eta(\rho, \lambda)(\mathcal{C})(v) = V_\lambda$, and this follows as a special case of equation (2.4) in a paper by Bouwer [1]. ∎

We now observe that the representation $\eta(\rho, \lambda)$ determines ρ in the following sense.

LEMMA 2. *Given two irreducible representations* (ρ_1, V_1) *and* (ρ_2, V_2) *of* L *such that there exists a linear functional* $\lambda \in \mathfrak{H}^*$ *with* $[\rho_1:\lambda]$ *and* $[\rho_2:\lambda] > 0$ *and* $\eta(\rho_1, \lambda)$ *equivalent to* $\eta(\rho_2, \lambda)$, *then* ρ_1 *is equivalent to* ρ_2.

PROOF. We first observe that if M is a maximal left ideal of $\mathcal{C}$ then there exists a unique maximal left ideal M' of U containing M. In fact, if M'' denotes the left ideal of U generated by M we can easily see that $M'' \neq U$ and hence the existence of M' is established by a simple application of Zorn's Lemma. For the uniqueness it suffices to show that the left regular representation $(\pi, U/M'')$ of U modulo M'' has a unique maximal $\pi(U)$-invariant subspace. Clearly $U/M'' = \sum \oplus (U/M'')_\gamma$ and if W is a $\pi(U)$-invariant subspace of U/M'' we have $W = \sum \oplus (W \cap (U/M'')_\gamma)$. Since $(U/M'')_\lambda$ is the representation space of an irreducible representation of $\mathcal{C}$ it follows that $W \cap (U/M'')_\lambda = (U/M'')_\lambda$ or $W \cap (U/M'')_\lambda = \{0\}$. In the first case $1 + M'' \in W$ and hence $W = U/M''$. Therefore every proper $\pi(U)$-invariant subspace of U/M'' is a subspace of $\sum_{\gamma \neq \lambda} \oplus (U/M'')_\gamma \neq U/M''$, and hence W_s, the sum of all proper $\pi(U)$-invariant subspaces of U/M'', is the unique maximal proper $\pi(U)$-invariant subspace of U/M''.

Next suppose M_1 and M_2 are two maximal left ideals of $\mathcal{C}$ such that the left regular representations of $\mathcal{C}$ modulo M_1 and $\mathcal{C}$ modulo M_2 are equivalent. Then we claim that the left regular representations of U modulo M_1' and U modulo M_2' are equivalent. Indeed by assumption we have a $\mathcal{C}$-linear map $\phi: \mathcal{C}/M_1 \to \mathcal{C}/M_2$. Let $\phi(1 + M_1) = x + M_2$ and define $\phi': U/M_1' \to U/M_2'$ by setting $\phi'(u + M_1') = ux + M_2'$. It is then readily checked that ϕ' establishes the required equivalence.

Finally assume (ρ_1, V_1) and (ρ_2, V_2) are two irreducible representations of L such that there exists a linear functional $\lambda \in \mathfrak{H}^*$ with $[\rho_i:\lambda] > 0$ and the representations $(\eta(\rho_1, \lambda), (V_1)_\lambda)$ and $(\eta(\rho_2, \lambda), (V_2)_\lambda)$ are equivalent. Choose nonzero elements $v_i \in (V_i)_\lambda$ and define $M_i = \{c \in \mathcal{C} \mid \eta(\rho_i, \lambda)(c) v_i = 0\}$ for $i = 1, 2$. Clearly M_i is a maximal left ideal in $\mathcal{C}$ and $(\eta(\rho_i, \lambda), (V_i)_\lambda)$ is equivalent to the left regular representation of $\mathcal{C}$ modulo M_i. By assumption then the left regular representations of $\mathcal{C}$ modulo M_1 and $\mathcal{C}$ modulo M_2 are equivalent. As above, this implies that the left regular representations of U modulo M_1' and U modulo M_2' are equivalent. Then, by the uniqueness of M_i' we have $M_i' = \{u \in U \mid \rho_i(u)v_i = 0\}$ and (ρ_i, V_i) is equivalent to the left regular representation of U modulo M_i'. Therefore (ρ_1, V_1) is equivalent to (ρ_2, V_2). ∎

The result of Lemma 2 suggests a closer look at the irreducible representations of $\mathcal{C}$.

DEFINITION 3. A nontrivial representation (η, W) of $\mathcal{C}$ is said to be λ-*weighted* for some linear functional $\lambda \in \mathcal{H}^*$ iff it is irreducible and $\eta(H - \lambda(H) \cdot 1) = 0$ for all $H \in \mathcal{H}$.

LEMMA 3. *Every λ-weighted representation of $\mathcal{C}$ is equivalent to a representation of the form $(\eta(\rho, \lambda), V_\lambda)$ for some irreducible representation (ρ, V) of L with $[\rho:\lambda] > 0$.*

PROOF. Let (η, W) be a λ-weighted representation of $\mathcal{C}$ and select a nonzero element $w \in W$. Set $M = \{c \in \mathcal{C} \mid \eta(c)w = 0\}$. Clearly M is a maximal left ideal of $\mathcal{C}$. Now we claim that the left regular representation of U modulo M' (the unique maximal left ideal of U containing M) is the required irreducible representation. This follows immediately on observing that $M' \cap \mathcal{C} = M$. To prove this latter fact we note that $M \subseteq M' \cap \mathcal{C}$ by definition of M'. Then take any $x \in M' \cap \mathcal{C}$ and assume $x \notin M$. By the maximality of M in $\mathcal{C}$ there exists an element $y \in \mathcal{C}$ such that $yx - 1 \in M$ and hence $1 \in M'$. This contradiction implies that $M = M' \cap \mathcal{C}$. ∎

Finally combining the results of this section we have

THEOREM 1. *For any fixed linear functional $\lambda \in \mathcal{H}^*$ there is a one-to-one correspondence between the set of equivalence classes of irreducible representations (ρ, V) of L with $[\rho:\lambda] > 0$ and the set of equivalence classes of λ-weighted representations of $\mathcal{C}$.*

2. **Relations between weight spaces.** Two natural questions now arise:

(1) Suppose (ρ, V) is an irreducible representation of L such that $[\rho:\lambda_1] > 0$ and $[\rho:\lambda_2] > 0$ for two different linear functionals $\lambda_1, \lambda_2 \in \mathcal{H}^*$. Then how are the representations $\eta(\rho, \lambda_1)$ and $\eta(\rho, \lambda_2)$ related?

(2) Conversely, let (η_i, W_i) be a λ_i-weighted representation of $\mathcal{C}$ for $i = 1, 2$. If $\lambda_1 \neq \lambda_2$, under what conditions does there exist a common irreducible representation (ρ, V) of L such that η_i is equivalent to $\eta(\rho, \lambda_i)$ for $i = 1, 2$?

Unfortunately we have been unable to provide a "$\mathcal{C}$-internal" answer to these questions; however, we do have the following straightforward result which we will use in the next section for somewhat more satisfactory results in a restricted case.

THEOREM 2. *Let* (η_i, W_i) *be a* λ_i*-weighted representation of* $\mathcal{C}$ *and let* M_i *denote a maximal left ideal of* $\mathcal{C}$ *such that* η_i *is equivalent to the left regular representation of* $\mathcal{C}$ *modulo* M_i *for* $i=1, 2$. *Then there exists an irreducible representation* (ρ, V) *of* L *such that* η_i *is equivalent to* $\eta(\rho, \lambda_i)$ $(i=1, 2)$ *iff there exists an element* $x \in U - M_2'$ *such that* $M_1' x \subseteq M_2'$.

PROOF. If there exists an element $x \in U - M_2'$ such that $M_1' x \subseteq M_2'$ then the map $\phi: U/M_1' \to U/M_2'$ defined by $\phi(u+M_1') = ux + M_2'$ is a linear isomorphism which establishes the equivalence between the left regular representation of U modulo M_1' and of U modulo M_2'. Let (ρ, V) be any representation in this equivalence class; it is clear that η_i is equivalent to $\eta(\rho, \lambda_i)$ of $i=1, 2$.

Conversely if there exists an irreducible representation (ρ, V) of L such that η_i is equivalent to $\eta(\rho, \lambda_i)$ then M_i can be considered to be the left annihilator ideal of some nonzero element $v_i \in V_{\lambda_i}$—i.e. $M_i = \{c \in \mathcal{C} \mid \rho(c)v_i = 0\}$—for $i=1, 2$. Moreover, as in §1 we have $M_i' = \{u \in U \mid \rho(u)v_i = 0\}$ for $i=1, 2$. Since ρ is assumped to be irreducible there exists an element $x \in U$ such that $\rho(x)v_2 = v_1$ and hence $x \notin M_2'$ and $M_1' x \subseteq M_2'$. ∎

3. **One-dimensional weight spaces.** In the case of all finite-dimensional irreducible representations or, more generally, all irreducible representations (ρ, V) of L admitting a "highest weight", it is well known that there always exists at least one linear functional $\lambda \in \mathcal{H}^*$ such that $[\rho : \lambda] = 1$. On the other hand, if (ρ, V) is an irreducible representation of L for which there exists a linear functional $\lambda \in \mathcal{H}^*$ with $[\rho : \lambda] = 1$ then, as above, ρ is determined by $\eta(\rho, \lambda)$ and moreover $\eta(\rho, \lambda)$ may be regarded as an algebra homomorphism from $\mathcal{C}$ into K. This particular class of representations of L was studied by the author in a previous paper [**3**]. Unfortunately, it is possible for inequivalent one-dimensional representations of $\mathcal{C}$ to yield equivalent representations of L. The next two theorems are aimed at shedding some light on the relationship between one-dimensional representations of $\mathcal{C}$ which yield equivalent representations of L.

THEOREM 3. *Let* (η_i, K) *be a* λ_i*-weighted one-dimensional representation of* $\mathcal{C}$ *for* $i=1, 2$. *If there exists an irreducible representation* (ρ, V) *of* L *such that* η_i *is equivalent to* $\eta(\rho, \lambda_i)$ *for* $i=1, 2$ *then there exist elements* $x, y \in U$ *such that* $yx \in \mathcal{C}$, $\eta_2(yx) = 1$ *and* $\eta_1(c) = \eta_2(ycx)$ *for all* $c \in \mathcal{C}$.

PROOF. Choose $0\neq v_i \in V_{\lambda_i}$. Then $M_i=\{c\in \mathcal{C} \mid \rho(c)v_i=0\}$ is a maximal left ideal of $\mathcal{C}$ such that η_i is equivalent to the left regular representation of $\mathcal{C}$ modulo M_i. By Theorem 2 there exists an element $x\in U - M_2'$ such that $M_1' x\subseteq M_2'$. Since $x\notin M_2'$ there exists an element $y\in U$ such that $yx-1\in M_2'$. Then clearly $yx\in\mathcal{C}$ and $\eta_2(yx)=1$. Finally, since $\eta_2(y(c-\eta_1(c)\cdot 1)x)=0$, $\eta_1(c)=\eta_2(ycx)$ for all $c\in\mathcal{C}$. ∎

As a partial converse to Theorem 3 we have:

THEOREM 4. *Let (η_i, K) be a one-dimensional λ_i-weighted representation of $\mathcal{C}$ and denote* $\mathrm{Ker}(\eta_i)$ *by M_i for $i=1, 2$. If there exists an element $x\in U-M_2'$ such that for all $y\in U$ satisfying $yx\in\mathcal{C}$ we have $\eta_2(ycx)=\eta_1(c)\eta_2(yx)$ for all $c\in\mathcal{C}$ then there exists an irreducible representation (ρ, V) of L such that η_i is equivalent to $\eta(\rho, \lambda_i)$ for $i=1, 2$.*

PROOF. We first observe that the set

$$\{u\in U \mid (\forall y\in U\colon yu\in\mathcal{C})yu\in M_i\}$$

is a maximal left ideal of U containing M_i and hence is equal to M_i'.

Let $(\pi, U/M_2')$ denote the left regular representation of U modulo M_2'. For all $c\in\mathcal{C}$ we have $cx\equiv\eta_1(c)x$ mod M_2'. Suppose, to the contrary, that $cx-\eta_1(c)x\notin M_2'$. By maximality of M_2', there exists an element $y\in U$ such that $y(cx-\eta_1(c)x)-1\in M_2'$—i.e. $\eta_2(ycx-\eta_1(c)yx)=1$. However, by assumption $\eta_2(ycx-\eta_1(c)yx)=0$. This contradiction implies that $cx-\eta_1(c)x\in M_2'$. Therefore η_1 is equivalent to $\eta(\pi, \lambda_1)$. ∎

4. Some interesting questions. Having established the close relationship between the representations of L and those of $\mathcal{C}$, we are now interested in looking at the irreducible representations of $\mathcal{C}$ and the structure of $\mathcal{C}$. In this regard we have far more questions than answers. In previous papers [3], [4] we have shown that $\mathcal{C}$ is a finitely generated subalgebra of U and if (η, W) is a finite-dimensional, λ-weighted representation of $\mathcal{C}$ then the associated irreducible representation (ρ, V) of L has the property that $[\rho:\lambda]<\infty$ for all $\lambda\in\mathcal{H}^*$.

From these observed facts we face the following questions:

(1) Are all λ-weighted (resp. irreducible) representations of $\mathcal{C}$ finite-dimensional?

(2) Are all irreducible representations of $\mathcal{C}$ λ-weighted for some linear functional $\lambda\in\mathcal{H}^*$?

(3) If (η, W) is a finite-dimensional λ_0-weighted representation of L, does there exist a linear functional $\lambda\in\mathcal{H}^*$ such that $[\rho:\lambda]=1$?

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