ON THE RELATION BETWEEN THE ABEL AND BOREL-TYPE METHODS OF SUMMABILITY

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1. Introduction. It is known that the Abel method and the Borel exponential method of summability are not equivalent, but that under certain conditions, both methods sum the same series to the same sum [5]. This was recently extended in one direction, to the conditions under which a series summable by a Borel-type method is also summable by the Abel method [7]. The object of this paper is to extend this last result to absolute summability.

2. Definitions and generalities. Suppose throughout that \( a_n \) \((n = 0, 1, \cdots)\) are arbitrary complex numbers, that \( a > 0 \) and that \( \beta \) is real. Let \( N \) be any nonnegative integer greater than \( 1 - \beta/a \). Let \( M \) denote a positive constant, not necessarily the same at each occurrence.

Define

\[
\begin{align*}
  s_n &= \sum_{r=0}^{n} a_r; \quad s_{-1} = 0; \quad \sigma_N = \sigma - s_{N-1}.
\end{align*}
\]

2.1. Definitions of the Borel-type methods of summability.

Define

\[
\begin{align*}
  \phi(x) &= \sum_{n=0}^{\infty} a_n x^{an + \beta - 1};
  \psi(x) = \sum_{n=0}^{\infty} s_n x^{an + \beta - 1}
\end{align*}
\]

It is known [1] that the convergence of one of these series for all \( x \geq 0 \) implies the convergence of the other for all \( x \geq 0 \); henceforth it is assumed that these series are convergent for all \( x \geq 0 \).

Define \( S(x) = S_{a,\beta}(x) = ae^{-x} \psi(x); A(x) = \int_{0}^{x} e^{-t} a(t) dt \).

Note. Except in the lemma in §4, the suffixed form \( S_{a,\beta}(x) \) will not be used.

Ordinary summability [2]. If \( S(x) \to \sigma \) as \( x \to \infty \), then \( s_n \to \sigma(B, \alpha, \beta) \).

If \( A(x) \to \sigma_N \) as \( x \to \infty \), then \( s_n \to \sigma(B', \alpha, \beta) \).

Absolute summability [4]. If \( s_n \to \sigma(B, \alpha, \beta) \) and \( S(x) \) is of bounded variation with respect to \( x \) on the interval \([0, \infty)\) then \( s_n \to \sigma \mid B \mid, \alpha, \beta \). If \( s_n \to \sigma(B', \alpha, \beta) \) and \( A(x) \) is of bounded variation with respect to \( x \) on the interval \([0, \infty)\), then \( s_n \to \sigma \mid B' \mid, \alpha, \beta \).

Received by the editors August 30, 1968.

1 The research for this paper was supported in part by the National Research Council.
Note. The summability method \((B, 1, 1)\) is the classical Borel exponential method \((B)\) and the method \((B', 1, 1)\), the classical Borel integral method \((B')\).

The actual choice of \(N\) is immaterial. Thus \(N\) will henceforth be assumed to be sufficiently large so that \(S(0) = 0\) and \(x^{1-\beta}S'(x)\) is continuous for \(x \geq 0\). Further, it may be assumed without loss of generality that \(a_0 = a_1 = \cdots = a_{N-1} = 0\), so that \(\sigma_N = \sigma\).

2.2. Definitions of the Abel methods of summability. Define

\[
L(x) = \sum_{n=0}^{\infty} a_n x^n = (1 - x) \sum_{n=0}^{\infty} s_n x^n.
\]

Ordinary summability \([6, p. 7]\). If \(L(x) = \sum_{n=0}^{\infty} a_n x^n\) is convergent for \(|x| < 1\) and \(L(x) \to \sigma\) as \(x \to 1 -\), then \(s_n \to \sigma(A)\).

Absolute summability \([8]\). If \(s_n \to \sigma(A)\) and \(L(x)\) is of bounded variation with respect to \(x\) on the interval \([0, 1)\), then \(s_n \to \sigma|A|\).

3. Theorems for ordinary methods. In 1931, Doetsch \([5]\) proved the following theorem:

**Theorem A.** If \(s_n \to \sigma(B)\) and \(L(x) = \sum_{n=0}^{\infty} a_n x^n\) is convergent for \(|x| < 1\), then \(s_n \to \sigma(A)\).

This was extended in 1961 by Jajte \([7]\) to give

**Theorem B.** If \(s_n \to \sigma(C, k)(B, \alpha, \beta)\) where \(0 \leq k \leq 1\), and \(L(x) = \sum_{n=0}^{\infty} a_n x^n\) is convergent for \(|x| < 1\), then \(s_n \to \sigma(A)\).

Note. \((C, 0)(B, \alpha, \beta)\) is the \((B, \alpha, \beta)\) method applied to the \((C, k)\) mean of \(s_n\) (the Cesaro mean of order \(k\) \([6, p. 96]\)).

Since \((C, 0)\) is convergence, the relation between the Borel-type method and the Abel methods is expressed as

**Corollary B.** If \(s_n \to \sigma(B, \alpha, \beta)\) and \(L(x) = \sum_{n=0}^{\infty} a_n x^n\) is convergent for \(|x| < 1\), then \(s_n \to \sigma(A)\).

Since it is known \([3, Theorem 2]\) that \(s_n \to \sigma(B, \alpha, \beta)\) if and only if \(s_n \to \sigma(B', \alpha, \beta - 1)\), the following theorem is immediate:

**Theorem 1.** If \(s_n \to \sigma(B', \alpha, \beta)\) and \(L(x) = \sum_{n=0}^{\infty} a_n x^n\) is convergent for \(|x| < 1\), then \(s_n \to \sigma(A)\).

4. Theorems for absolute methods. In this section, Corollary B and Theorem 1 are extended to absolute summability.

**Theorem 2.** If \(s_n \to \sigma|B, \alpha, \beta|\) and \(L(x) = \sum_{n=0}^{\infty} a_n x^n\) is convergent for \(|x| < 1\), then \(s_n \to \sigma|A|\).
Proof. Because of Corollary B and since \( S(x), L(x) \) are absolutely continuous on \([0, \infty), [0, 1)\) respectively, it is sufficient for the proof of Theorem 2 to prove that

\[
\int_0^1 |L'(t)| \, dt < \infty \quad \text{whenever} \quad \int_0^\infty |S'(t)| \, dt < \infty.
\]

The following lemma is required:

Lemma. If \( s_n \to \sigma |B, \alpha, \beta| \) then \( s_n \to \sigma |B, \alpha, \beta + \delta| \) whenever \( \delta > 0 \).

Proof. (Note. In this proof, the suffixed form \( S_{a, \beta}(x) \) is used.) Since it is known that \( s_n \to \sigma (B, \alpha, \beta + \delta) \) whenever \( s_n \to \sigma (B, \alpha, \beta) \) and \( \delta > 0 \) [3, Result II], it suffices to show that

\[
\int_0^\infty |S_{a, \beta} (l) | \, dl < \infty \quad \text{whenever} \quad \int_0^\infty |S'_a (l) | \, dl < \infty.
\]

Thus, since [4, Result I]

\[
\Gamma (\delta) S_{a, \beta} (l) = e^{-l} \int_0^l (t - u)^{l-1} e^u S'(u) \, du,
\]

it follows that

\[
\Gamma (\delta) \int_0^\infty |S_{a, \beta} (l) | \, dl = \int_0^\infty e^{-l} \, dt \int_0^l (t - u)^{l-1} e^u |S'(u) | \, du
\]

\[
= \int_0^\infty e^u |S'(u) | \, du \int_0^\infty (t - u)^{l-1} e^{-t} \, dt
\]

\[
= \Gamma (\delta) \int_0^\infty |S_{a, \beta} (u) | \, du < \infty.
\]

This completes the proof of the lemma.

The direct proof of Corollary B consists of taking the Laplace transform of \( S(x) \) and knowing that whenever \( S(x) \to \sigma \) as \( x \to \infty \),

\[
I(y) = y \int_0^\infty e^{-yu} S(u) \, du \to \sigma \quad \text{as} \quad y \to 0+,
\]

\[
B(y) = (1 + y)^{\beta - \alpha} \left \{ \frac{(1 + y)^\alpha - 1}{\alpha y} \right \} \to 1 \quad \text{as} \quad y \to 0+,
\]

and \( L(x) = B(y) I(y) \) where \( x \) and \( y \) are related by \( x = (1 + y)^{-\alpha} \).

Note. This relation between \( x \) and \( y \) is assumed implicitly for the remainder of this proof.
First, note that
\[ I(y) = \int_0^\infty e^{-yu} S'(u) \, du \quad \text{and} \quad I'(y) = -\int_0^\infty e^{-yu} u S'(u) \, du. \]

In order to show that \( L(x) \) is of bounded variation with respect to \( x \) on the interval \([0, 1)\), it is sufficient to prove that
\[ \int_0^\infty \left| \frac{d}{dy} B(y) I(y) \right| \, dy < \infty. \]

Now, note the following properties of \( B(y) \):
(i) \( B(y) \to 1 \) as \( y \to 0^+ \).
(ii) \( B(y) \) is continuous for \( y > 0 \).
(iii) \( B(y) \sim y^{\beta-1/\alpha} \) as \( y \to \infty \).

Also, for \( y > 0 \)
\[
\frac{B'(y)}{B(y)} = \frac{\beta - \alpha - 1}{1 + y} + \frac{\alpha(1 + y)^{\alpha-1}}{(1 + y)^\alpha - 1}.
\]

Thus, \( B'(y) \) has the following properties:
(iv) \( B'(y) \to (2\beta - \alpha - 1)/2 \) as \( y \to 0^+ \).
(v) \( B'(y) \) is continuous for \( y > 0 \).
(vi) \( B'(y) \sim (\beta - 1)y^{\beta-1/\alpha} \) as \( y \to \infty \).

In view of all these properties, since \( \beta > 1 \) and since \( t^{\alpha-1} S'(t) \) is continuous for \( t \geq 0 \), it now follows that

(a) \[
\int_1^\infty \left| B'(y) I(y) \right| \, dy \leq \int_1^\infty M y^\beta - 2 dy \int_0^\infty e^{-yu} \left| S'(u) \right| \, du
\]
\[ = M \int_0^\infty \left| S'(u) \right| \, du \int_1^\infty y^\beta - 2 e^{-yu} \, dy
\]
\[ = M \int_0^\infty u^{\alpha-\beta} \left| S'(u) \right| \, du < \infty, \]

(b) \[
\int_1^\infty \left| B(y) I'(y) \right| \, dy \leq \int_1^\infty M y^{\beta - 1} dy \int_0^\infty u e^{-yu} \left| S'(u) \right| \, du
\]
\[ = M \int_0^\infty u \left| S'(u) \right| \, du \int_1^\infty y^{\beta - 1} e^{-yu} \, dy
\]
\[ = M \int_0^\infty u^{\alpha-\beta} \left| S'(u) \right| \, du < \infty, \]
(c) \[ \int_0^1 |B(y)I'(y)| \, dy \leq \int_0^1 M \int_0^\infty u e^{-u} |S'(u)| \, du \]
\[ = M \int_0^\infty u |S'(u)| \, du \int_0^1 e^{-u} \, dy \]
\[ = M \int_0^\infty (1 - e^{-u}) |S'(u)| \, du < \infty, \]

and

(d) \[ \int_0^1 |B'(y)I(y)| \, dy < \infty \]

since \( B'(y) \) and \( I(y) \) are bounded on \([0, 1]\).

Thus, it follows from (a), (b), (c) and (d), that

\[ \int_0^\infty \left| \frac{d}{dy} B(y)I(y) \right| \, dy < \infty, \]

and this completes the proof of Theorem 2.

Since it is known that \( s_n \rightarrow \sigma [B, \alpha, \beta] \) if and only if \( s_n \rightarrow \sigma [B', \alpha, \beta - 1] \) [4, Theorem 17], the following theorem follows immediately:

**Theorem 3.** If \( s_n \rightarrow \sigma [B', \alpha, \beta] \) and \( L(x) = \sum_{n=0}^\infty a_n x^n \) is convergent for \( |x| < 1 \), then \( s_n \rightarrow \sigma [A] \).

**References**


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