ON THE HAUSDORFF OPEN CONTINUOUS IMAGES OF
HAUSDORFF PARACOMPACT \( p \)-SPACES

H. H. WICKE\(^1\)

1. Introduction. Ponomarev proved the following remarkable theorem: Every \( T_0 \) first-countable space of infinite cardinality is an open continuous image of a zero-dimensional metrizable space of the same weight [8].\(^2\) This theorem clearly and succinctly summarizes the behavior of metrizable spaces under open mappings. The purpose of this article is to prove an analogue of Ponomarev's theorem in a not necessarily first-countable situation and to develop some of its consequences. This analogue, Theorem 1 below, is a joint discovery of the author and Dr. J. M. Worrell, Jr. [10]. Remark 4 shows how a proof of Ponomarev's theorem may be derived from the proof of Theorem 1. Theorem 1 leads directly to a characterization (Theorem 2) of the class of Hausdorff open continuous images of Hausdorff paracompact \( p \)-spaces as the class of Hausdorff spaces of point-countable type. The latter class generalizes the class of Hausdorff first-countable spaces. Both the concept of \( p \)-space and of space of point-countable type are due to Arhangel'skiï [3], [4]. Theorem 3, a rather direct consequence of Theorem 1, answers a question of Arhangel'skiï by generalizing a theorem of his to the Hausdorff case. A relation between Theorem 1, which involves single-valued mappings, and Theorem 3, which involves many-valued mappings, is pointed out in Remark 3.

2. Terminology. The general terminology used here is much like that of [7], one exception being that spaces called compact in [7] are here called bicompact. The usage of [7] in letting \( X \) ambiguously denote the topological space \( (X, \tau) \) is followed where convenient, and product space refers to a Cartesian product of spaces endowed with the product topology [7]. A base for \( X \) means a base for the topology of \( X \). The letter \( N \) denotes the set of positive integers and if \( A \) is a set, \( \aleph(A) \) denotes the cardinal number of \( A \). The weight [2] of a topological space \( (X, \tau) \) is defined as the smallest cardinal number

---

\(^1\) This work was supported by the United States Atomic Energy Commission.

\(^2\) Ponomarev does not point out that infinite cardinality is required. In fact, if \( S \) is a finite \( T_0 \) but not \( T_1 \) space any \( T_1 \) open continuous preimage of \( S \) has infinite weight and cardinality. In Theorem 1 infinite cardinality is not required since the spaces here are assumed to be \( T_1 \).
ber \( m \) such that \( \mathcal{B} \) has a base of cardinal \( m \). A mapping \( f: X \to Y \) is called \textit{perfect} \([1]\) if and only if it is closed, continuous, and \( f^{-1}(y) \) is bicompact for all \( y \in Y \). If \( \mathcal{A} \) is a collection of sets then \( \text{St}(x, \mathcal{A}) \) denotes \( \cup \{ A \in \mathcal{A} : x \in A \} \). A \( T_1 \)-space \( X \) is called a \( p \)-space \([3]\) if and only if there exists a sequence \( g_1, g_2, \ldots \) of collections of open subsets of the Wallman bicompactification \( \omega X \) of \( X \) covering \( X \) such that if \( x \in X, \cap \{ \text{St}(x, g_n) : n \in \mathbb{N} \} \subseteq X \). If \( X \) is a Tychonoff space this definition is equivalent to one in which \( \beta X \) (the Stone-Cech bicompactification of \( X \)) replaces \( \omega X \). A principal theorem for \( p \)-spaces, suggestive of the naturality of their use in Theorem 1, is that of Arhangel'skii: A \( T_2 \)-space is a paracompact \( p \)-space if and only if there exists a perfect mapping of it onto a metrizable space \([3, \text{Theorem 5.1}]\).

3. \textbf{Theorems.} If \((X, \mathcal{A})\) is a space and \( A \subseteq X \), a subcollection \( \mathcal{D} \) of \( \mathcal{B} \) whose members include \( A \) is called a \textit{base at} \( A \) if and only if for every \( U \in \mathcal{B} \) such that \( U \supseteq A \), there exists \( D \in \mathcal{D} \) such that \( A \subseteq D \subseteq U \).

If \( X \) is a space and \( A \subseteq X \), then \( A \) is said to be of \textit{countable character} \([4]\) if and only if there exists a countable base at \( A \).

A space \( X \) is said to be of \textit{point-countable type} \([4]\) if and only if \( X \) is covered by a collection of bicompact subspaces of countable character.

Remark 1. Any first-countable space is of point-countable type.

Remark 2. The property of being of point-countable type is preserved by open continuous mappings.

The following lemma was stated by Arhangel'skii \([5, \text{p. 158}]\). A proof is sketched here for completeness.

\textbf{Lemma 1.} A Tychonoff \( p \)-space is of point-countable type.

\textbf{Proof.} Every point of such a space \( X \) lies in a bicompact subset of \( X \) which is a \( G_\delta \)-set in \( \beta X \) and every such set has countable character.

\textbf{Lemma 2.} In a Hausdorff space \( X \) the following properties are equivalent:

(i) \( X \) is of point-countable type.

(ii) If \( U \) is open in \( X \) and \( x \in U \) there exists a bicompact set \( B \) of countable character such that \( x \in B \subseteq U \).

\textbf{Proof.} Clearly (ii) implies (i). Suppose \( x \in U \) and \( U \) is open. There exists a bicompact set \( B \) of countable character containing \( x \). Let \( \{ U_k : k \in \mathbb{N} \} \) be a base at \( B \) such that \( U_{k+1} \subseteq U_k \) for all \( k \in \mathbb{N} \). Then since \( X \) is Hausdorff \( B = \cap \{ \overline{U}_k : k \in \mathbb{N} \} \). Let \( V_1 = U \). Suppose open sets \( V_1, \ldots, V_n \) have been defined such that \( x \in V_k \subseteq U_k \cap V_{k-1} \) and \( \overline{V}_k \) is disjoint from \( B \sim V_{k-1} \) for \( 1 < k \leq n \). Since \( B \sim V_n \) is bicompact, \( x \in V_n \) and \( X \) is \( T_2 \), there exists an open set \( V \) such that \( x \in V \subseteq \overline{V} \subseteq X \sim (B \sim V_n) \). Let \( V_{n+1} = V \cap V_n \cap U_{n+1} \). Thus there exists a sequence
\{ V_n \} such that for all \( n \in \mathbb{N} \), \( x \in V_{n+1} \subseteq V_n \cap U_{n+1} \) and \( V_{n+1} \) is disjoint from \( B \sim V_n \). Let \( C = \bigcap \{ V_n : n \in \mathbb{N} \} \). Then \( C \) is a closed (therefore bicom pact) subset of \( B \) containing \( x \). Since \( V_{n+1} \subseteq (X \sim B) \cup V_n \), \( C = \bigcap \{ V_n : n \in \mathbb{N} \} \) and \( C \subseteq U. \) Suppose \( W \) is open and \( C \subseteq W. \) If no \( V_n \subseteq W \), there exists a sequence \( \{ x_k \} \) such that each \( x_k \in V_n \sim W. \) Since \( \bigcap \{ V_k \sim W : k \in \mathbb{N} \} = \emptyset \) and \( B \) is bicom pact, there exists \( n \) such that \( V_k \sim W \subseteq X \sim B \) for all \( k \geq n \). Let \( A = \{ x_k : k \geq n \} \). Then \( A \subseteq X \sim W \) and \( \bar{A} \cap B \neq \emptyset \). For if \( B \subseteq X \sim \bar{A} \), then for some \( k \geq n \), \( U_k \subseteq X \sim \bar{A} \subseteq X \sim A \) contradicting \( x_k \in A. \) If \( y \in \bar{A} \cap B \), \( y \in \overline{V_k} \sim W \) for all \( k \in \mathbb{N} \), again a contradiction. Hence some \( V_n \subseteq W \), so that \( C \) has countable character.

**Theorem 1.** Suppose \( X \) is a Hausdorff space of point-countable type. Then \( X \) is the range of an open continuous mapping \( \phi \) such that: (1) The domain \( Y \) of \( \phi \) is a Hausdorff paracompact p-space. (2) The weight of \( Y \) is the weight of \( X \). (3) \( Y \) is a subspace of the product space of a zero-dimensional metrizable space and \( X \).

**Proof.** See §4.

**Comment.** For Tychonoff spaces, part (1) can be derived from [4, Theorem 3.14] by the method of Remark 3 below.

**Theorem 2.** A Hausdorff space is of point-countable type if and only if it is an open continuous image of a Hausdorff paracompact p-space.

**Proof.** This follows from Theorem 1, Lemma 1, and Remark 2.

Recall that a many-valued mapping \( f : X \rightarrow Y \) is called continuous (from above) [9] if and only if for every \( x \in X \) if \( V \subseteq Y \) is open and \( f(x) \subseteq V \) there exists an open \( U \subseteq X \) such that \( x \in U \) and \( f(U) \subseteq V \). The mapping \( f \) is called range-bicom pact (or \( Y \)-bicom pact [9]) if and only if \( f(x) \) is bicom pact for every \( x \in X \). Arhangel’skii proved the following theorem with the additional hypothesis that \( X \) is a Tychonoff space [4, Theorem 3.14] and asked [4, p. 54] whether it is valid for a wider class of spaces.

**Theorem 3.** Suppose \( X \) is a Hausdorff space. Then \( X \) is of point-countable type if and only if \( X \) is the range of an open continuous (possibly many-valued) range-bicom pact mapping of a metrizable space.

**Proof.** By Theorem 1, there exists a continuous mapping \( \phi \) of a \( T_2 \) paracompact p-space \( Y \) onto \( X \). By Arhangel’skii’s theorem (see §2) there exists a perfect mapping \( \theta \) of \( Y \) onto a metrizable space \( M \). It is straightforward to show that \( \phi \circ \theta^{-1} \) is an open continuous range-bicom pact mapping of \( M \) onto \( X \). The sufficiency follows from [4, Proposition 3.6].
Remark 3. Theorem 3 can be used to derive part (1) of Theorem 1. For if $f$ is an open continuous many-valued range-bicompact mapping of a metrizable space $X$ onto a Hausdorff space $Y$ of point-countable type, let $Z = \{(x, y) \in X \times Y : y \in f(x)\}$, under the topology induced by the product topology. The set $Z$ is called the graph of $f$ by Ponomarev [9]. If $\theta$ and $\phi$ denote the projections of $Z$ onto $X$ and $Y$ respectively, then it may be seen that $f = \phi \circ \theta^{-1}$ where $\phi$ is open and continuous and $\theta$ is perfect. (This statement may be proved in a fashion similar to that used by Ponomarev in showing that a perfect mapping $f$ factors into $\phi \circ \theta^{-1}$ where $\theta$ and $\phi$ are perfect [9, Theorem 1, §2].) Hence $Z$ is a paracompact $p$-space by Arhangel’skii's theorem and $\phi$ maps $Z$ onto $Y$.

4. Proof of Theorem 1.

Proof. Assume $\mathcal{N}(X)$ is infinite. Let $\mathcal{C}$ denote $\{B \subset X : B$ is bicompact and of countable character$\}$. For some base $\mathcal{W}$ of $X$ such that weight of $X = \mathcal{N}(\mathcal{W})$, let $\mathcal{F}$ denote the collection of all unions of finite subcollections of $\mathcal{W}$. Then $\mathcal{N}(\mathcal{F}) = \text{weight of } X$ and $\mathcal{W} \subset \mathcal{F}$. Call a sequence $\alpha$ admissible if and only if for each $n \in N$: (1) $\alpha(n) \in \mathcal{F}$; (2) $\alpha(n+1) \subset \alpha(n)$; (3) for some $B \in \mathcal{C}$, $B = \bigcap \{\alpha(k) : k \in N\}$ and $\{\alpha(k) : k \in N\}$ is a base at $B$. Using bicompactness it may be seen that for each $B \in \mathcal{C}$ there exists an admissible sequence $\alpha$ satisfying (3) with respect to $B$.

Consider $\mathcal{F}$ as a topological space with the discrete topology and let $\Delta$ denote the product space of countably many copies of $\mathcal{F}$. Let $\Gamma = \{\alpha \in \Delta : \alpha$ is admissible$\}$. Then $\Gamma$ is a metrizable zero-dimensional space (it is a subspace of a Baire space [6]). Let $\Gamma \times X$ denote the product space of $\Gamma$ and $X$ and let

$$Y = \{(\alpha, x) \in \Gamma \times X : x \in \cap \{\alpha(k) : k \in N\}\},$$

with the topology induced by the product topology. Note that $Y$ is Hausdorff. Let $\theta = \pi_1 | Y$ and $\phi = \pi_2 | Y$, where $\pi_i$ denotes projection onto the $i$th coordinate. Then $\theta$ and $\phi$ are continuous mappings of $Y$ onto $\Gamma$ and $X$ respectively.

If $\alpha \in \Gamma$, let $S(\alpha | n) = \{\alpha' \in \Gamma : \alpha'(k) = \alpha(k), k = 1, \ldots, n\}$. Then $\{S(\alpha | n) : n \in N$ and $\alpha \in \Gamma\}$ is a base for $\Gamma$. For $\alpha \in \Gamma$ and $V \in \mathcal{F}$ such that $V \subset \alpha(n)$ let $D(\alpha | n ; V)$ denote $(S(\alpha | n) \times V) \cap Y$. Then $\mathcal{B} = \{D(\alpha | n ; V) : \alpha \in \Gamma, n \in N, V \in \mathcal{F}, \text{ and } V \subset \alpha(n)\}$ is a base for $Y$. Since $\mathcal{N}(\mathcal{F}) = \text{weight of } X$, $\mathcal{N}(\mathcal{B}) = \text{weight of } X$.

Suppose $\alpha \in \Gamma$, $V \in \mathcal{F}$, and $V \subset \alpha(n)$. Then clearly $\phi[D(\alpha | n ; V)] \subset V$. If $x \in V$, then by Lemma 2 there exists $B \in \mathcal{C}$ such that $x \in B \subset V$. Let $\beta \in \Gamma$ be such that $\{\beta(k) : k \in N\}$ is a base at $B$. There exists $k$ such that $\beta(k) \subset V$. The sequence $\alpha'$ such that $\alpha'(j) = \alpha(j), 1 \leq j \leq n$
and \( \alpha'(j) = \beta(k + j) \) for \( j > n \) is admissible and \( (\alpha', x) \in D(\alpha| n; V) \). Hence \( \phi[D(\alpha| n; V)] = V \). Therefore \( \phi \) is an open mapping.

If it is shown that \( \theta \) is a perfect mapping, then by Arhangel'skiï's theorem cited in §2, \( Y \) is a paracompact \( p \)-space. Suppose \( \alpha \in \Gamma \) and \( B = \cap \{ \alpha(k): k \in N \} \). Then, since \( B \) is bicompact, \( \theta^{-1}(\alpha) = \{ \alpha \} \times B \) is bicompact. Hence \( \theta \) is a bicompact mapping. To show that \( \theta \) is closed suppose \( W \) is open in \( Y \) and \( \theta^{-1}(\alpha) \subset W \). There exist \( m \in N \) and sets \( D_k = D(\alpha_k| n(k); V_k) \in \Theta \) intersecting \( \theta^{-1}(\alpha) \) for \( k = 1, \ldots, m \), such that \( \theta^{-1}(\alpha) \subset \bigcup \{ D_k: k \leq m \} \subset W \). Since \( \theta^{-1}(\alpha) \) meets each \( D_k \), \( \alpha_k(j) = \alpha(j), 1 \leq j \leq n(k), 1 \leq k \leq m \). Also \( B \subset \bigcup \{ V_k: k \leq m \} \). By conditions (2) and (3) on admissible sequences there exists \( n \geq \max \{ n(k): k \leq m \} \) such that \( B \subset \alpha(n) \subset \bigcup \{ V_k: k \leq m \} \). If \( (\alpha', x) \in D = D(\alpha| n); (\alpha(n)) \), then \( x \in V_k \) for some \( k \), and therefore \( (\alpha', x) \in D_k \subset W \). Hence \( \theta^{-1}(\alpha) \subset D \subset W \). Since any \( \theta^{-1}(\alpha') \) intersecting \( D \) is a subset of \( D \), \( D = \theta^{-1}(\theta(D)) \). It follows that \( \theta \) is a closed mapping.

Remark 4. If the space \( X \) is \( T_0 \) and first-countable, then \( \Theta \) in the above proof can be taken as the collection \( \{ \{ x \}: x \in X \} \). Then each admissible sequence \( \alpha \) is such that \( \cap \{ \alpha(k): k \in N \} = \{ x \} \) for some \( x \in X \). It follows that \( Y \) is homeomorphic to \( \Gamma \) and thus \( X \) is an open continuous image of \( \Gamma \). This proves Ponomarev's theorem.

REFERENCES


3. A. V. Arhangel'skiï, On a class of spaces containing all metric and all locally bicompact spaces, Mat. Sb. 67 (1965), 55–85. (Russian)


Sandia Laboratories, Albuquerque, New Mexico