LEBESGUE CHARACTERIZATIONS OF UNIFORMITY-DIMENSION FUNCTIONS

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1. Introduction. Let \((X, p)\) be a metric space, let \(\text{dim}(X)\) be the covering dimension of \(X\), and let \(d_0(X, p)\) be the metric dimension of \(X\). Let \(d_2\) and \(d_3\) denote the metric-dependent dimension functions introduced by Nagami and Roberts \([7]\), and let \(d_6\) and \(d_7\) be the metric-dependent dimension functions introduced by Smith \([9]\). Characterizations of some of these metric-dependent dimension functions in terms of Lebesgue covers have been given by Egorov \([1]\), Wilkinson \([11]\) and Smith \([9]\). These results are described by the following table.

<table>
<thead>
<tr>
<th>Metric-dependent dimension function</th>
<th>Characterization</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_4(X, p)\leq n)</td>
<td>Every Lebesgue cover consisting of (n+2) members has an open refinement of order (\leq n+1)</td>
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<tr>
<td>(d_6(X, p)\leq n)</td>
<td>Every finite Lebesgue cover has an open refinement of order (\leq n+1)</td>
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<tr>
<td>(d_8(X, p)\leq n)</td>
<td>Every countable Lebesgue cover has an open refinement of order (\leq n+1)</td>
</tr>
<tr>
<td>(d_0(X, p)\leq n)</td>
<td>Every locally finite Lebesgue cover has an open refinement of order (\leq n+1)</td>
</tr>
<tr>
<td>(d_5(X, p)\leq n)</td>
<td>Every Lebesgue cover has an open refinement of order (\leq n+1)</td>
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</table>

Soniat \([10]\) has generalized the dimension functions \(d_0\), \(d_2\) and \(d_3\) for uniform spaces and obtained Lebesgue-type characterizations for \(d_4\) and \(d_6\). In this paper we complete the above characterization table for uniform spaces. In §2 we develop Lebesgue cover properties for uniform spaces and characterize \(d_2\). In §§3 and 4 we generalize the dimension functions \(d_6\) and \(d_7\) to uniform spaces and characterize them in terms of Lebesgue covers.

**Definition.** Let \(X\) be a set and \(\mathcal{D} = \{D_\lambda : \lambda \in A\}\) be a family of collections of subsets of \(X\). For each \(\lambda \in A\), let \(\mathcal{D}_\lambda = \{D_\alpha : \alpha \in A_\lambda\}\). Then

\[
\bigwedge_{\lambda \in A} \{D_\lambda\} = \{\bigcap D_{\alpha(\lambda)} : \alpha(\lambda) \in A_\lambda, \lambda \in A\}.
\]

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Throughout this paper \( J \) will denote the set \( \{1, 2, \cdots, n+1\} \) and \( J'=J\cup\{n+2\} \), where the integer \( n \) will always be understood.

2. Characterization of \( d_2 \) for uniform spaces. The reader is referred to the papers by Nagami and Roberts [7], Smith [9], and Soniat [10] for the definitions of the dimension functions \( d_0, d_2, d_3, d_6 \) and \( d_7 \) and the generalizations of \( d_0, d_2, \) and \( d_3 \) to uniform spaces.

**Definition 2.1.** Let \( C \) and \( C' \) be subsets of a uniform space \((X, \mathcal{U})\). We say that \( C \) and \( C' \) are separated provided there exists \( U\in\mathcal{U} \) such that \((C\times C')\cap U=\emptyset\). If \( \mathcal{E} = \{C_\alpha, C'_\alpha : \alpha \in A\} \) is a family of pairs \((C_\alpha, C'_\alpha)\), then \( \mathcal{E} \) is called uniformly separated if there exists \( U\in\mathcal{U} \) such that \((C_\alpha\times C'_\alpha)\cap U=\emptyset \) for all \( \alpha \in A \).

**Definition 2.2.** A cover \( \mathcal{D} \) of a uniform space \((X, \mathcal{U})\) is called Lebesgue if there exists \( U\in\mathcal{U} \) such that \( \{U(x) : x \in X\} \) refines \( \mathcal{D} \).

**Definition 2.3.** A cover \( \mathcal{D} = \{D_\alpha : \alpha \in A\} \) of a uniform space \((X, \mathcal{U})\) is called \( \mathcal{U} \)-shrinkable if there exists some \( U\in\mathcal{U} \) and a cover \( \mathcal{T} = \{T_\alpha : \alpha \in A\} \) such that

1. \( T_\alpha \subset D_\alpha \) for all \( \alpha \in A \).
2. \( \{T_\alpha, X-D_\alpha : \alpha \in A\} \) is uniformly separated by \( U \).

**Theorem 2.4.** A cover \( \mathcal{D} \) of a uniform space \((X, \mathcal{U})\) is Lebesgue if and only if \( \mathcal{D} \) is \( \mathcal{U} \)-shrinkable.

**Proof (Necessity).** Let \( \mathcal{D} = \{D_\alpha : \alpha \in A\} \) be a Lebesgue cover of \((X, \mathcal{U})\). Then there exists \( U\in\mathcal{U} \) such that \( \{U(x) : x \in X\} \) refines \( \mathcal{D} \). Choose \( V\in\mathcal{U} \) such that \( V \) is symmetric and \( V^2 \subset U \). Define \( F_\alpha = \{x : V(x) \cap (X-D_\alpha) = \emptyset\} \) for all \( \alpha \in A \).

(i) We assert that \( \{F_\alpha : \alpha \in A\} \) covers \( X \). Clearly \( F_\alpha \subset D_\alpha \) for all \( \alpha \in A \). Let \( x \in D_\alpha - F_\alpha \). Then since \( \{U(x) : x \in X\} \) refines \( \mathcal{D} \), there exists \( \beta \in A \) such that \( V(x) \subset U(x) \subset D_\beta \). Hence \( V(x) \cap (X-D_\beta) = \emptyset \) so that \( x \in F_\beta \).

(ii) We now assert that \( \{F_\alpha, X-D_\alpha : \alpha \in A\} \) is uniformly separated by \( V \). Suppose there exists some \( \beta \in A \) such that \( [F_\beta \times (X-D_\beta)] \cap V \neq \emptyset \). Let \( (x, y) \in [F_\beta \times (X-D_\beta)] \cap V \). Then \( x \in F_\beta \), \( y \in X-D_\beta \) and \( X \in V(y) \) and \( y \in V(x) \). But \( x \in F_\beta \) implies \( V(x) \cap [X-D_\beta] = \emptyset \) so \( y \notin V(x) \), a contradiction.

**Remark.** It should be noted at this point that the cover \( \mathcal{F} = \{F_\alpha : \alpha \in A\} \) defined above is Lebesgue. If \( x \in X \), then there exists \( \beta \in A \) such that \( x \in U(x) \subset D_\beta \). Clearly \( x \in F_\beta \) and we assert that \( V(x) \subset F_\beta \). Let \( y \in V(x) \) and \( z \in V(y) \), so that \( (x, y) \in V \) and \( (y, z) \in V \). Hence \( (x, z) \in V^2 \subset U \) and therefore \( z \in U(x) \subset D_\beta \). It now follows that \( V(y) \cap (X-D_\beta) = \emptyset \) and \( y \in F_\beta \). Thus \( V(x) \subset F_\beta \).
(Sufficiency). Suppose $\mathcal{D} = \{D_\alpha : \alpha \in \mathcal{A}\}$ is $\mathcal{U}$-shrinkable to $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$, where $\{F_\alpha, X - D_\alpha : \alpha \in \mathcal{A}\}$ is uniformly separated by symmetric $U \subseteq \mathcal{U}$. Let $x \in X$. Since $\mathcal{F}$ is a cover of $X$, there exists $\beta \in \mathcal{A}$ such that $x \in F_\beta$. Let $y \in U(x)$ so that $(x, y) \in U$. But

$$U \cap [F_\beta \times (X - D_\beta)] = \emptyset$$

and hence $y \in X - D_\beta$. Thus $y \in D_\beta$ and $U(x) \subseteq D_\beta$. Therefore

$$\{U(x) : x \in X\}$$

refines $\mathcal{D}$ and $\mathcal{D}$ is Lebesgue.

For normal uniform spaces $(X, \mathcal{U})$ Soniat has shown the following [10, Theorem 3.8].

**Theorem 2.5.** Let $(X, \mathcal{U})$ be a normal uniform space. Then $d_2(X, \mathcal{U}) \leq n$ if and only if for every uniformly separated collection $\{C_i, C'_i : i \in J\}$ of closed sets, there exists a closed collection $\{B_i : i \in J\}$ such that $B_i$ separates $C_i$ and $C'_i$ and $\bigcap_{i \in J} B_i = \emptyset$.

**Note.** Here the separating sets $B_i$ are subsets of $X$ and separate $C_i$ and $C'_i$ in the usual sense and are not to be confused with elements of the uniformity $\mathcal{U}$.

The next theorem now follows directly from [9, Theorem 2.3] where the Lebesgue covers are now in the uniformity sense rather than the metric sense.

**Theorem 2.6.** Let $(X, \mathcal{U})$ be a completely normal uniform space. Then $d_2(X, \mathcal{U}) \leq n$ if and only if for every collection $\{D_i : i \in J\}$ of $n + 1$ binary Lebesgue covers of $X$, the cover $\mathcal{D} = \Lambda_{i \in J} D_i$ of $X$ has an open refinement of order $\leq n + 1$.

We now obtain a Lebesgue characterization of $(X, \mathcal{U})$ analogous to [9, Theorem 2.4].

**Theorem 2.7.** Let $(X, \mathcal{U})$ be a completely normal uniform space. Then $d_2(X, \mathcal{U}) \leq n$ if and only if every Lebesgue cover $\mathcal{D} = \{D_1, D_2, \ldots, D_{n+2}\}$ of $X$ consisting of $n + 2$ members has an open refinement of order $\leq n + 1$.

**Proof (Necessity).** Suppose $d_2(X, \mathcal{U}) \leq n$, and let $\mathcal{D} = \{D_1, D_2, \ldots, D_{n+2}\}$ be a Lebesgue cover of $X$. Then there exists $U \subseteq \mathcal{U}$ such that $\{U(x) : x \in X\}$ refines $\mathcal{D}$. By Theorem 2.4 above we can uniformly shrink $\mathcal{D}$ to a closed Lebesgue cover $\mathcal{F} = \{F_1, F_2, \ldots, F_{n+2}\}$ such that $F_i \subseteq D_i$ for $i \in J'$. Then for each $i \in J$, $D_i = \{D_i, X - F_i\}$ is a binary Lebesgue cover of $X$. By Theorem 2.6 above $\mathcal{D}^* = \Lambda_{i \in J} D_i$ has an open refinement $\mathcal{D}^{**}$ such that $\text{ord}(\mathcal{D}^{**}) \leq n + 1$. But $\mathcal{D}^*$ refines $\mathcal{D}$ since $\mathcal{F}$ covers $X$. Hence $\mathcal{D}^{**}$ is the desired open cover.
(Sufficiency). Let \( \{ C_i, C_i' : i \in J \} \) be a collection of \( n+1 \) pairs of closed sets which are uniformly separated by \( U \subseteq \mathcal{U} \). Choose \( K \) and \( V \) symmetric in \( \mathcal{U} \) such that \( K \subseteq K_i \subseteq \forall \subseteq V \subseteq U \). Now define \( \mathcal{K} = \{ K(x) : x \in X \} \) and \( \mathcal{V} = \{ V(x) : x \in X \} \). Define for each \( i \in J \), \( D_i = \text{St}(C_i, \mathcal{V}) \) and \( H_i = \text{St}(C_i, \mathcal{K}) \) where \( \text{St}(C_i, \mathcal{K}) \) is the star of \( C_i \) with respect to the cover \( \mathcal{K} \). Let \( D_{n+2} = X - \bigcup_{i \in J} H_i \). Clearly \( \Delta = \{ D_1, D_2, \ldots, D_{n+2} \} \) is an open Lebesgue cover of \( X \). Hence \( \Delta \) has an open refinement \( \mathcal{A} = \{ R_\alpha : \alpha \in A \} \) such that the \( \text{ord}(\mathcal{A}) \leq n+1 \). Define \( f \) to be the function, \( f : A \rightarrow J' \), such that

\[
f(\alpha) = \{ \text{smallest integer } i \in J' \text{ such that } R_\alpha \subseteq D_i \}.
\]

Now define \( \mathcal{R}_i = \bigcup \{ R_\alpha : i = \alpha \} \) for each \( i \in J' \). Hence \( \mathcal{A} = \{ R_1, R_2, \ldots, R_{n+2} \} \) may replace \( \{ R_\alpha : \alpha \in A \} \). Choose \( K^* \in \mathcal{K} \) such that \( (K^*)^2 \subseteq K \) and define

\[
\mathcal{K}^* = \{ K^*(x) : x \in X \}, \quad E_i = \{ x : x \in C_i, x \in R_i \},
\]

\[
S_i = \text{St}(E_i, \mathcal{K}^*), \quad R_i^* = R_i \cup S_i, \quad \text{for } i \in J, \quad \text{and } R_{n+2}^* = R_{n+2}.
\]

Now \( S_i \cap D_{n+2} = \emptyset \); for \( x \in S_i \) implies that \( x \in \text{St}(E_i, \mathcal{K}^*) \subseteq \text{St}(C_i, \mathcal{K}) \subseteq H_i \) so that \( x \notin D_{n+2} \). Hence \( \mathcal{A}^* = \{ R_1^*, R_2^*, \ldots, R_{n+2}^* \} \) is an open cover of \( X \) such that \( \text{ord}(\mathcal{A}^*) \leq n+1 \) and \( C_i \subseteq R_i^* \) for \( i \in J \).

Since \( \mathcal{A}^* \) is finite there exists by [5, Lemma 1.5] a closed cover \( \mathcal{F} = \{ F_1, F_2, \ldots, F_{n+2} \} \) of \( X \) such that \( C_i \subseteq F_i \subseteq R_i^* \) for \( i \in J \), and \( F_{n+2} \subseteq R_{n+2}^* \). X normal implies that there exist open sets \( 0_i \) such that \( F_i \subseteq 0_i \subseteq \text{int} R_i^* \) for \( i \in J \). Define \( B_i = 0_i - 0_i \) for \( i \in J \). Clearly \( B_i \) is a closed set separating \( C_i \) and \( C_i' \) for \( i \in J \). We assert \( \bigcap_{i \in J} B_i = \emptyset \). Suppose there exists \( x \in \bigcap_{i \in J} B_i \). Then \( x \in F_i \) for each \( i \in J \). Hence \( x \in F_{n+2} \subseteq R_{n+2}^* \). But \( x \in R_i^* \) for all \( i \in J \) and hence \( x \in \bigcap_{i \in J} R_i^* \). This is a contradiction since \( \text{ord}(\mathcal{A}^*) \leq n+1 \). Hence \( d_2(X, \mathcal{U}) \leq n \).

3. The uniformity dimension function \( d_2 \)

**Definition 3.1.** Let \((X, \mathcal{U})\) be a uniform space. If \( X = \emptyset \), \( d_2(X, \mathcal{U}) = -1 \). Otherwise \( d_2(X, \mathcal{U}) \leq n \) if \((X, \mathcal{U})\) satisfies this condition:

\( (D_2) \) Given any countable collection of closed pairs \( \{ C_i, C_i' : i = 1, 2, \ldots \} \) such that

1. \( \{ C_i, C_i' : i = 1, 2, \ldots \} \) is uniformly separated,
2. \( \{ X - C_i : i = 1, 2, \ldots \} \) is locally finite,

then there exist closed sets \( B_i \) separating \( C_i \) and \( C_i' \) such that \( \text{ord}(B_i : i = 1, 2, \ldots) \leq n \).

**Theorem 3.2.** Let \((X, \mathcal{U})\) be a paracompact uniform space. Then \( d_2(X, \mathcal{U}) \leq n \) if and only if every countable, locally finite Lebesgue cover of \( X \) has an open refinement of order \( \leq n+1 \).
Proof. The proof is essentially the same proof as that of [9, Theorem 3.2].

Theorem 3.3. Let \((X, \mathcal{U})\) be a uniform space. Then every countable Lebesgue cover of \(X\) has a countable locally finite Lebesgue refinement.

Proof. Let \(\mathcal{D} = \{D_1, D_2, \ldots \}\) be a Lebesgue cover of \(X\). Then there exists \(U \in \mathcal{U}\) such that \(\{U(x) : x \in X\}\) refines \(\mathcal{D}\). Choose \(V\) and \(K\) symmetric in \(\mathcal{U}\) such that \(K \subseteq K' \subseteq V \subseteq V' \subseteq U\) and define \(F_i = \{x : V(x) \cap [X - D_i] = \emptyset\}\) for all \(i\). As before \(\mathcal{F} = \{F_i : i = 1, 2, \ldots \}\) is a Lebesgue cover of \(X\). Now let

\[ R_i = D_i - \bigcup_{j < i} \text{St}(F_j, K) \], where \(K = \{K(x) : x \in X\}\).

Clearly \(\mathcal{R} = \{R_1, R_2, \ldots \}\) refines \(\mathcal{D}\) in a 1–1 manner. We assert that \(\mathcal{R}\) is a locally finite Lebesgue cover of \(X\).

(i) Let \(x \in X\). Choose the smallest \(i\) such that \(x \in \text{St}(F_i, K)\). Then \(x \in D_i - \bigcup_{j < i} \text{St}(F_j, K) = R_i\) and hence \(R\) covers \(X\). Also \(\text{St}(F_i, K) \cap R_i = \emptyset\) for all \(i > j\) so that \(\mathcal{R}\) is locally finite.

(ii) Let \(x \in X\). Choose the smallest \(i\) such that \(K(x) \cap \text{St}(F_i, K) \neq \emptyset\). Clearly \(K(x) \subseteq X - \bigcup_{j < i} \text{St}(F_j, K)\). We claim that \(K(x) \subseteq D_i\).

Let \(y \in K(x)\). Since \(K(x)\) can be open, there exists \(r \in X\) such that \(r \in K(x) \cap \text{St}(F_i, K)\). Thus there exist \(s \in X\) and \(t \in F_i\) such that \(r \in K(s)\) and \(t \in K(s) \cap F_i\). Therefore we have \(t \in K(s), s \in K(r), r \in K(x)\), and \(x \in K(y)\). Hence \((t, y) \in K' \subseteq V\) so that \(y \in V(t)\). By definition \(t \in F_i\) implies that \(V(t) \cap [X - D_i] = \emptyset\). Thus \(y \in V(t) \subseteq D_i\), so that \(K(x) \subseteq D_i\).

We now have a Lebesgue characterization for \(d_\mathcal{U}\).

Theorem 3.4. Let \((X, \mathcal{U})\) be a paracompact uniform space. Then \(d_\mathcal{U}(X, \mathcal{U}) \leq n\) if and only if every countable Lebesgue cover has an open refinement of order \(\leq n + 1\).

4. The uniformity dimension function \(d_\mathcal{U}\)

Definition 4.1. Let \((X, \mathcal{U})\) be a uniform space. If \(X = \emptyset\), then \(d_\mathcal{U}(X, \mathcal{U}) = -1\). Otherwise, \(d_\mathcal{U}(X, \mathcal{U}) \leq n\) if \((X, \mathcal{U})\) satisfies this condition.

\((D_7)\) Given any collection of closed pairs \(\{C_\alpha, C'_\alpha : \alpha \in A\}\) such that

1. \(\{C_\alpha, C'_\alpha : \alpha \in A\}\) are uniformly separated.
2. \(\{X - C_\alpha : \alpha \in A\}\) is locally finite.

Then there exist closed sets \(B_\alpha\) separating \(C_\alpha\) and \(C'_\alpha\) such that \(\operatorname{ord} \{B_\alpha : \alpha \in A\} \leq n\).
Theorem 4.2. Let \((X, \mathcal{U})\) be a paracompact uniform space. Then 
\[ d_1(X, \mathcal{U}) \leq n \text{ if and only if every locally finite Lebesgue cover of } X \text{ has a refinement of order } \leq n + 1. \]

Proof. The proof is essentially the same as proof of [9, Theorem 4.2]. Paracompactness is now required so that [6, Theorem 1.3] is applicable.

5. Conclusion. The table in paragraph 1 is now complete for uniform spaces \((X, \mathcal{U})\). The Lebesgue characterizations are exactly the same as for metric spaces but complete normality is required for the dimension functions \(d_2, d_3\) and paracompactness is required for \(d_4, d_5\) and \(d_6\).

References


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