ON CENTRALIZERS OF INVOLUTIONS

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1. Introduction. The main purpose of this paper is to establish sufficient conditions for a group of even order to contain a normal elementary Abelian 2-subgroup of order at most 4 (Theorem 1). As a consequence it is shown that PSL(2, 5) is the only simple group which contains an involution \( x \) with the following property: the Sylow 2-subgroup of the centralizer \( C \) of \( x \) in \( G \) is a noncyclic group of order 4 which is normal in \( C \) (Theorem 3).

Several corollaries are derived from Theorem 1. In particular, a direct proof is given of the fact that PSL(2, 5) is the only group which has no normal 2-complement, no normal elementary Abelian 2-subgroups of order less than 8 and which contains an involution with an elementary Abelian centralizer of order 4 (Theorem 2).

If \( G \) is a group, \( x \in G \) and \( T \) is a subset of \( G \), \( C_G(x), Cl_G(x), I(T), o(T), o(x), \langle T \rangle, T^*, Z(G) \) and \( K(G) \) denote respectively: the centralizer of \( x \) in \( G \), the conjugate class of \( x \) in \( G \), the set of involutions in \( T \), the number of elements in \( T \), the order of \( x \), the group generated by \( T \), \( T - \{1\} \), the center of \( G \) and the largest normal subgroup of \( G \) of odd order. If \( P \) is a \( p \)-group then \( O_1(P) \) is the subgroup of \( P \) generated by elements of \( P \) of order \( p \).

From now on \( G \) will be a group of even order, \( x \) a fixed involution of \( G \), \( K = K(G), C = C_G(x), I = I(C_G(x)), Cl(x) = Cl_G(x) \), and \( S \) a fixed Sylow 2-subgroup of \( G \) containing \( x \) such that \( S_0 = S \cap C = \text{Sylow 2-subgroup of} \ C \). We are ready to state the results.

**Theorem 1.** Suppose that there exists \( y \in I - Cl(x) \) such that

\[
(*) \quad C_G(u) \cap Cl_G(y) \subseteq Cl_G(y)
\]

for all \( u \in I \). Then \( \langle Cl_G(y) \rangle \) is a proper elementary Abelian normal 2-subgroup of \( G \).

If, in addition, \( I \cap \langle Cl_G(y) \rangle = \{y\} \), then \( o(\langle Cl_G(y) \rangle) \leq 4 \).

**Corollary 1.** Suppose that the following conditions hold:

(a) \( I = I(C_G(u)) \) for all \( u \in Cl(x) \cap I \);

(b) \( I(C_G(y)) = I(C_G(z)) \) for all \( y, z \in I - Cl(x) \). Then one of the following statements holds.

(i) \( G \) has one class of involutions and \( \langle I \rangle \) is an elementary Abelian normal 2-subgroup of \( C \).

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(ii) $G$ has at least two classes of involutions and it contains a proper elementary Abelian normal 2-subgroup.

**Corollary 2.** Suppose that $o(I) \leq 3$. Then one of the following statements holds.

(i) $S_0 = S$, $x$ is the only involution in $S$ and $\langle x \rangle K$ is a normal subgroup of $G$.

(ii) $S_0 = S$, $S$ contains exactly 3 involutions and $\langle x \rangle K$ is a proper normal subgroup of $G$.

(iii) $S_0 = S$, $G$ has one conjugate class of involutions.

(iv) $G$ has at least 2 classes of involutions and it contains a normal elementary Abelian subgroup of order at most 4.

Corollary 2 immediately yields

**Corollary 3.** Suppose that $o(I)^3$ and $G$ is simple. Then $S = S_0$ and $G$ has only one conjugate class of involutions.

In case that $C$ is elementary Abelian of order 4 we get the following

**Theorem 2.** Suppose that $C = \{1, x, y, xy\}$ is elementary Abelian and $G$ has neither a normal 2-complement nor a normal elementary Abelian 2-subgroup of order less than 8. Then $G \cong \text{PSL}(2, 5)$.

The following corollary is an easy consequence of Theorem 2, the results of Suzuki in [6] and the results of Feit and Thompson in [2].

**Corollary 4.** Let $G$ be a finite noncyclic simple group containing an element $w$ such that $o(C_0(w)) \leq 4$. Then $G$ is isomorphic to one of the following groups: $\text{PSL}(2, 5)$, $\text{PSL}(2, 7)$, $A_6$ and $A_7$.

Our final theorem requires the deep results of Gorenstein and Walter [5] with respect to groups with a dihedral Sylow subgroup of order 4.

**Theorem 3.** Suppose that $S_0 = \{1, x, y, xy\}$ is elementary Abelian, $S_0$ is normal in $C$ and $G$ is simple. Then $G \cong \text{PSL}(2, 5)$.

The proof of Theorem 1 utilizes the following lemma, which is of independent interest.

**Lemma.** Let $U$ be a subgroup of the group $H$ and let $w$ be an involution of $H$ which normalizes $U$ leaving fixed exactly two elements of $U$, 1 and $y$. Let $V$ be a normal, $w$-invariant noncyclic elementary Abelian subgroup of $U$ containing $y$. Then $V$ is a Sylow 2-subgroup of $U$, $o(V) = 4$, and $U/V$ is Abelian.
2. Proof of the Lemma, Theorem 1 and Corollary 1. We begin with the proof of the Lemma. Obviously $y$ is an involution. First assume that $o(V) = 4$, $V = \{1, y, z, yz\}$; then $z^w = yz$. Suppose that $U/V$ is not an Abelian group of odd order. Then $w$ fixes an element of $(U/V)^*$, say $uV$. Thus one of the following holds:

$$u^w = uy \quad \text{and} \quad u = u^{w^2} = u$$

$$= uz \quad = uy$$

$$= uyz \quad = uy.$$

Hence we must have $u^w = uy$; but then $(uz)^w = (uy)(yz) = uz$ a contradiction. Thus $U/V$ is an Abelian group of odd order. If $o(V) > 4$, then $w$ fixes an element of $(V/(y))^*$, say $z(y)$, and $V_0 = \langle z, y \rangle$ is a normal, $w$-invariant, elementary Abelian subgroup of $V$ containing $y$, $o(V_0) = 4$, and by the first part $V = V_0$, a contradiction. The proof of the Lemma is complete.

To prove Theorem 1, suppose first that $Cl_G(y) \nsubseteq C_G(y)$ and let $t \in Cl_G(y) - C_G(y)$. By a result of Brauer and Fowler [1, p. 572], there exists $w \in I(G)$ such that $w \in I(C_G(x)) \cap C_G(t) \subseteq I$. Hence by (*) $t \in C_G(w) \cap Cl_G(y) \subseteq C_G(y)$ a contradiction. It follows that $Cl_G(y) \subseteq C_G(y)$ and $(Cl_G(y)) = H$ is a normal subgroup of $G$ contained in $C_G(y)$. If $C_G(y) = G$, then $H = \langle y \rangle \neq G$ and the theorem follows. If $C_G(y) \neq G$, then $H$ is a proper normal subgroup of $G$ and obviously $y \in \Omega_1(P) < G$ where $P$ is the Sylow 2-subgroup of $Z(H)$. Hence $Cl_G(y) \subseteq \Omega_1(P)$ and $H$ is elementary Abelian. Finally suppose that $o(H) \geq 8$ and $I \cap H = \{y\}$. Then $x$ leaves only $y$ and $1$ fixed in $H$ and by the Lemma $o(H) = 4$, a contradiction. Thus $o(H) \leq 4$ and the proof of Theorem 1 is complete.

It remains to prove Corollary 1. If $I \subseteq Cl(x)$, then each element of $I$ belongs to the center of some Sylow 2-subgroup of $G$ and therefore $G$ has one class of involutions. By (a), (I) is an elementary Abelian normal 2-subgroup of $C$ and (i) holds. Suppose finally that $I \subseteq Cl(x)$ and let $y \in I - Cl(x)$. It follows from (b) that the elements of $I - Cl(x)$ commute with each other. Thus for all $u \in I \cap Cl(x)$,

$$C_G(u) \cap Cl_G(y) = I \cap Cl_G(y) \subseteq C_G(y),$$

and for all $u \in I - Cl(x)$,

$$C_G(u) \cap Cl_G(y) = I(C_G(y)) \cap Cl_G(y) \subseteq C_G(y).$$

It follows then by Theorem 1 that $G$ has a proper normal elementary Abelian 2-subgroup.
3. Proof of Theorem 2 and Corollaries 2 and 4. We begin with Corollary 2. If \( o(I) = 1 \), then \( S_0 = S \), \( x \) is the only involution in \( S \) and by \([3]\), \( \langle x \rangle K \) is a normal subgroup of \( G \), as described in (i). As \( o(I) \neq 2 \), let \( o(I) = 3 \), \( I = \{ x, y, xy \} \). If no element of \( I \) is conjugate to \( x \) in \( G \), then \( N_S(S_0) = S_0 \), \( S = S_0 \), and by \([3]\) \( \langle x \rangle K \triangleleft G \). Since \( o(I) = 3 \), \( \langle x \rangle K \neq G \) and (ii) holds. If all the elements of \( I \) are conjugate in \( G \), then again \( S_0 = S \) and (iii) holds. Suppose finally that \( x \) is conjugate to \( xy \) in \( G \), but not to \( y \). Then \( I(C_\sigma(xy)) = I \) and by Corollary 1, \( \langle Cl_\sigma(y) \rangle \) is a normal elementary Abelian 2-subgroup of \( G \). Hence, as either \( \langle Cl_\sigma(y) \rangle = \{ y \} \) or \( Cl_\sigma(y) \) contains an element which does not commute with \( x \), \( I \cap (Cl_\sigma(y)) = \{ y \} \) and by Theorem 1, \( o(\langle Cl_\sigma(y) \rangle) \leq 4 \), so that (iv) holds. This completes the proof of Corollary 2.

We continue with Theorem 2. If \( C = S \), then by Lemma 15.2.4 of \([4]\), \( G \) has only one class of involutions and \( N = N_\sigma(C) \cong PSL(2, 3) \). Thus \( C \) contains the centralizer of each of its nonunit elements and by Theorem 9.3.2 in \([4]\), due to Suzuki, \( G \) is a Zassenhaus group of degree 5 with \( N \) the subgroup fixing a letter. Thus \( N \) is a Frobenius group with complement of order \( e = 3 \) and kernel of order \( n = 4 \). Since \( e \) is odd and \( e = n - 1 \), it follows from Theorems 13.3.5 and 13.1.1 in \([4]\), due to Zassenhaus, that \( G \cong PSL(2, 4) \cong PSL(2, 5) \). Next assume that \( C \neq S \) and let \( y \in C \cap Z(S) \). As \( N_S(C) \neq C \), \( xy \) is conjugate to \( x \) in \( G \) and \( C_\sigma(xy) = C \). Since \( y \) is not conjugate to \( x \) in \( G \), it follows from Theorem 1 that \( \langle Cl_\sigma(y) \rangle \) is a normal elementary Abelian 2-subgroup of \( G \). As before \( I \cap (Cl_\sigma(y)) = \{ y \} \), and it follows by Theorem 1 that \( o(\langle Cl_\sigma(y) \rangle) \leq 4 \) in contradiction to our assumptions. The proof is complete.

It remains to prove Corollary 4. If \( o(C_\sigma(w)) = 2 \), then \( G \) is not simple. If \( o(C_\sigma(w)) = 3 \), then by \([2]\), \( G \) is isomorphic either to \( PSL(2, 5) \) or to \( PSL(2, 7) \). If \( o(C_\sigma(w)) = 4 \) and \( o(w) = 4 \), then by \([6]\), \( G \) is isomorphic to one of the groups \( PSL(2, 7) \), \( A_6 \) and \( A_7 \). If, finally, \( o(C_\sigma(w)) = 4 \) and \( o(w) = 2 \), then by Theorem 2, \( G \cong PSL(2, 5) \).

4. Proof of Theorem 3. If \( S = S_0 \), then by \([5]\), \( G \cong PSL(2, q), q > 3 \). If \( q \) is even, then \( G \cong PSL(2, 4) \cong PSL(2, 5) \). If \( q \) is odd, then the centralizer \( C \) of an involution of \( G \) is a dihedral group of order \( q + \epsilon \), \( \epsilon = \pm 1 \). For \( S \) to be normal in \( C \), \( q + \epsilon = 4 \) and \( q = 5 \). Thus again \( G \cong PSL(2, 5) \). Suppose next that \( S_0 \neq S \), \( \{ y \} = Z(S) \cap S_0 \). Then \( N_S(S_0) \neq S_0 \), \( xy \) is conjugate to \( x \) in \( G \) and \( S_0 \) is the normal Sylow 2-subgroup of \( C_\sigma(xy) \). As \( y \) is not conjugate to \( x \) in \( G \), it follows from Corollary 1 that \( G \) contains a proper, nontrivial, normal subgroup, in contradiction to the simplicity of \( G \). The proof is complete.
References


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