COMPACT MEANS IN THE PLANE

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1. **Results.** An *n-mean* ($n \geqq 2$) is a nonvoid Hausdorff space X together with a continuous symmetric idempotent function (which is also called an n-mean) from X^n into X. A space on which an n-mean can be defined is called an m_n-space [**2**, p. 210]. In the present note we show that a compact m_n-space embedded in the cartesian plane $\boldsymbol{R}^2$ does not separate $\boldsymbol{R}^2$ and, as a partial converse, that any compact locally connected subset of $\boldsymbol{R}^2$ that does not separate $\boldsymbol{R}^2$ is an m_n-space.

These two theorems reduce the problem of characterizing compact m_n-spaces in $\boldsymbol{R}^2$ to the question: When is a nonvoid compact non-locally connected subset of $\boldsymbol{R}^2$ with connected complement an m_n-space? The question is not answered here, but the answer cannot be either "always" or "never," for, on the one hand, there is a compact subset of $\boldsymbol{R}^2$ with connected complement that is not an m_2-space [**3**] and, on the other hand, a semilattice described in [**1**, p. 185, Example 1] is a 2-mean on a compact connected subset of $\boldsymbol{R}^2$ that is not locally connected.

2. **Proofs.** Our lemmas concern Čech homology theory on the category of compact pairs. We establish our notation with the following remarks.

Suppose G is an abelian group, X is a finite complex and A is a subcomplex of X. The G-valued n-chains of the oriented n-simplexes of X that assume the value 0 on the oriented n-simplexes of A form, under functional addition, an abelian group which will be denoted by $C_n(X, A; G)$. The boundary operator ∂ is defined in the usual manner [**7**, p. 111] and

$Z_n(X, A; G)$ is the kernel of ∂: $C_n(X, A; G) \rightarrow C_{n-1}(X, A; G)$;

$B_n(X, A; G)$ is the image of ∂: $C_{n+1}(X, A; G) \rightarrow C_n(X, A; G)$; and

$H_n(X, A; G) = Z_n(X, A; G)/B_n(X, A; G)$.

For each integer n, a simplicial map f: $(X, A) \rightarrow (Y, B)$ induces a homomorphism $C_n(f)$: $C_n(X, A; G) \rightarrow C_n(Y, B; G)$ which in turn induces a homomorphism $H_n(f)$: $H_n(X, A; G) \rightarrow H_n(Y, B; G)$. If X is a compact Hausdorff space, $\mathrm{Cov}(X)$ is the set of all finite open covers of X. If K is in $\mathrm{Cov}(X)$, X_K is the nerve of K. If (X, A) is a compact pair (that is, X is compact and Hausdorff and A is a closed subset of X), if K, $J \in \mathrm{Cov}(X)$ and if J refines K, there is a unique projection

Received by the editors October 1, 1968.

$$\pi(K, J): H_n(X_J, A_J; G) \to H_n(X_K, A_K; G).$$

The inverse limit group defined by all such projections is the nth *Čech homology group for* (X, A) *with coefficients in* G and is denoted by $\boldsymbol{H}_n(X, A; G)$. If $\boldsymbol{h}$ is in $\boldsymbol{H}_n(X, A; G)$ and if K is in $\mathrm{Cov}(X)$, the Kth coordinate of $\boldsymbol{h}$ is $\boldsymbol{h}_K$. A continuous map f: $(X, A) \to (Y, B)$ between compact pairs induces a homomorphism f_*: $\boldsymbol{H}_n(X, A; G) \to \boldsymbol{H}_n(Y, B; G)$.

(2.1) *Suppose X is a connected compact Hausdorff space, $p \in X$ and n is an integer $\geqq 2$. Let maps d^n, f^{nj}: $X \to X^n$ ($j \in \{1, \cdots, n\}$) be defined by the formulas*

$$(d^n x)_i = x;$$
$$(f^{nj}x)_i = x, \qquad i = j,$$
$$= p, \qquad i \neq j;$$

$x \in X$, $i \in \{1, \cdots, n\}$. If G is an abelian group and $\boldsymbol{h} \in \boldsymbol{H}_1(X; G)$, then $d^n_* \boldsymbol{h} = \sum_{j=1}^n f^{nj}_* \boldsymbol{h}$.

PROOF. We first consider the case $n=2$. Suppose $E \in \mathrm{Cov}(X^2)$. Since X is compact, there is a J in $\mathrm{Cov}(X)$ such that

$$K = \{U \times V: U, V \in J\}$$

refines E. Let P be a member of J that contains p. Define simplicial maps

$$F^{21}, F^{22}, D^2: X_J \to (X^2)_K$$

by the rules $F^{21}U = U \times P$, $F^{22}U = P \times U$ and $D^2 U = U \times U$ for all U in J. Observe that $f^{21}U \subset F^{21}U$, $f^{22}U \subset F^{22}U$ and $d^2 U \subset D^2 U$ whenever U is in J. Suppose $g \in G$, $m \geqq 3$ and z in $Z_1(X_J; G)$ is such that

$$z = g \sum_{i=1}^{m} V_{i-1} V_i, \tag{1}$$

where $V_0 = V_m = P$ and $V_{i-1} \neq V_i$ whenever $i \in \{1, \cdots, m\}$. Define $w \in C_2((X^2)_K; G)$ by

$$w = g\Bigg[\sum_{i=1}^{n-1} \sum_{j=1}^{i} (V_i \times V_j)(V_{i+1} \times V_j)(V_{i+1} \times V_{j+1}) + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} (V_i \times V_j)(V_{i+1} \times V_{j+1})(V_i \times V_{j+1}) \Bigg].$$

Then direct computation shows that

$$C_1(F^{21})(z) + C_1(F^{22})(z) - C_1(D^2)(z) = \partial w. \tag{2}$$

Since X_J is connected, every cycle in $Z_1(X_J;\, G)$ is a finite sum of cycles of form (1). Hence for any z in $Z_1(X_J;\, G)$ there is a w in $C_2((X^2)_K;\, G)$ such that (2) holds. It follows immediately that, if $h \in H_1(X_J;\, G)$,

$$H_1(F^{21})(h) + H_1(F^{22})(h) - H_1(D^2)(h) = 0.$$

Now suppose $\boldsymbol{h} \in \boldsymbol{H}_1(X;\, G)$. Then

$$\begin{aligned}(\overset{21}{f_*}\boldsymbol{h} + \overset{22}{f_*}\boldsymbol{h} - \overset{2}{d_*}\boldsymbol{h})_E &= \pi(E, K)\left[(\overset{21}{f_*}\boldsymbol{h})_K + (\overset{22}{f_*}\boldsymbol{h})_K - (\overset{2}{d_*}\boldsymbol{h})_K\right] \\ &= \pi(E, K)\left[H_1(F^{21})(\boldsymbol{h}_J) + H_1(F^{22})(\boldsymbol{h}_J) - H_1(D^2)(\boldsymbol{h}_J)\right] \\ &= \pi(E, K)(0) = 0,\end{aligned}$$

which completes the proof for the case $n = 2$.

The proof for $n > 2$ proceeds by an induction on n. Let $\boldsymbol{h}$ be in $\boldsymbol{H}_1(X;\, G)$ and define maps $k\colon X^n \to X^{n+1}$ and $g\colon X^2 \to X^{n+1}$ by the formulas

$$k(x_1, \cdots, x_n) = (x_1, \cdots, x_n, x_n);$$
$$g(x_1, x_2) = (p, \cdots, p, x_1, x_2).$$

Using the case $n = 2$ we have $(kf^{nn})_*\boldsymbol{h} = (gd^2)_*\boldsymbol{h} = (gf^{21})_*\boldsymbol{h} + (gf^{22})_*\boldsymbol{h} = f_*^{n+1,n}\boldsymbol{h} + f_*^{n+1,n+1}\boldsymbol{h}$. Using this and the inductive hypothesis we have $d_*^{n+1}\boldsymbol{h} = (kd^n)_*\boldsymbol{h} = \sum_{j=1}^{n-1}(kf^{nj})_*\boldsymbol{h} + (kf^{nn})_*\boldsymbol{h} = \sum_{j=1}^{n+1}(kf^{nj})_*\boldsymbol{h}$.

Throughout the remainder of this paper $\boldsymbol{Z}_n$ denotes a cyclic group of order n.

(2.2) *Suppose X is a compact connected Hausdorff space, $p \in X$, n is an integer $\geqq 2$ and $m\colon X^n \to X$ is a continuous function such that if $x \in X$,*

(1) $m(x, x, \cdots, x) = x$;

(2) $m(x, p, \cdots, p) = m(p, x, p, \cdots, p) = \cdots = m(p, \cdots, p, x)$. *Then $\boldsymbol{H}_1(X;\, \boldsymbol{Z}_n) = 0$.*

Proof. We suppose the hypotheses of (2.2) and define f^{nj}, d^n: $X \to X^n$ as in the hypothesis of (2.1). Observe that md^n is the identity on X and that, for $j \in \{2, \cdots, n\}$, $mf^{nj} = mf^{n1}$. Let $\boldsymbol{h}$ be in $\boldsymbol{H}_1(X;\, \boldsymbol{Z}_n)$. By (2.1), $\boldsymbol{h} = m_* d_*^n \boldsymbol{h} = \sum_{j=1}^{n} m_* f_*^{nj}\boldsymbol{h} = n(m_* f_*^{n1}\boldsymbol{h}) = 0$, the last equality holding because the use of $\boldsymbol{Z}_n$ as coefficient group insures that every element of a Čech group is of order n.

(2.3) *Suppose $n \geqq 2$, X is a compact subset of $\boldsymbol{R}^n$ and G is a nontrivial abelian group for which $\boldsymbol{H}_{n-1}(X;\, G) = 0$. If the Čech homology theory* *on compact pairs using G as coefficient group is exact, then* $\boldsymbol{R}^n - X$ *is connected.*

PROOF. Suppose on the contrary $\boldsymbol{R}^n - X$ has a bounded component C. Let E be an n-ball with $X \cup C$ in its interior and its bounding $(n-1)$-sphere denoted by S. Define $T = E - C$. Since T is a proper subset of E and contains S, S is a retract of T. Accordingly the inclusion map $k: S \subset T$ induces a monomorphism $k_*: \boldsymbol{H}_{n-1}(S; G) \to \boldsymbol{H}_{n-1}(T; G)$ and since $\boldsymbol{H}_{n-1}(S; G) \approx G \neq 0$, $\boldsymbol{H}_{n-1}(T; G) \neq 0$. In the following commutative diagram the dimension preserving homomorphisms are induced by appropriate inclusion maps.

$$\begin{array}{ccccc} & & \boldsymbol{H}_{n-1}(X; G) & \overset{\partial_1}{\longleftarrow} & \boldsymbol{H}_n(X \cup C, X; G) \\ & & \downarrow v_* & & \downarrow u_* \\ \boldsymbol{H}_{n-1}(E; G) & \overset{j_*}{\longleftarrow} & \boldsymbol{H}_{n-1}(T; G) & \overset{\partial_2}{\longleftarrow} & \boldsymbol{H}_n(E, T; G). \end{array}$$

Since $\boldsymbol{H}_{n-1}(X; G) = 0$, $\partial_2 u_* = v_* \partial_1 = 0$. By [6, p. 266, Theorem 5.4], the excision u_* is an isomorphism. Hence ∂_2 is a 0-homomorphism and, by exactness, j_* is a monomorphism. Since $\boldsymbol{H}_{n-1}(T; G) \neq 0$, $\boldsymbol{H}_{n-1}(E; G) \neq 0$, which is false.

(2.4) *If a compact subset X of* $\boldsymbol{R}^2$ *is an* m_n*-space,* $\boldsymbol{R}^2 - X$ *is connected.*

PROOF. Suppose, on the contrary, that a compact subset X of $\boldsymbol{R}^2$ admits an n-mean m and separates a point p from a point q. $\boldsymbol{R}^2$ is locally connected and unicoherent [5, p. 73, Corollary 6]. By a theorem of A. H. Stone [8, p. 429, Theorem 1] some component C of X separates p from q. The restriction of m to C^n is an n-mean on C [2, p. 212, Satz 4]. By (2.2) $H_1(C, \boldsymbol{Z}_n) = 0$. The Čech homology theory for compact pairs with $\boldsymbol{Z}_n$ as coefficient group is exact [6, p. 248, Theorem 7.6]. By (2.3), $\boldsymbol{R}^2 - C$ is connected, which is a contradiction.

(2.5) *A nonvoid compact locally connected subset of* $\boldsymbol{R}^2$ *that does not separate* $\boldsymbol{R}^2$ *is an* m_n*-space.*

PROOF. Suppose X is a nonvoid compact locally connected subset of $\boldsymbol{R}^2$ and $\boldsymbol{R}^2 - X$ is connected. Consider first the case in which X is connected. The function $m: (\boldsymbol{R}^2)^n \to \boldsymbol{R}^2$ defined by $m(p_1, \cdots, p_n) = (1/n) \sum_{i=1}^n p_i$ shows that $\boldsymbol{R}^2$ is an m_n-space. Since X is a retract of $\boldsymbol{R}^2$ [4, p. 132, (13.1)] and any retract of an m_n-space is an m_n-space [2, p. 212, Satz 3], X is an m_n-space.

If X is not connected, X has at most finitely many components, $C_1, \cdots, C_k$, each of which, by the above argument, is an m_n-space. For each i in $\{1, \cdots, k\}$ let $m_i: C_i^n \to C_i$ be an n-mean. Let p be a point of X. We define an n-mean $m\colon X^n \to X$ by the rule

$$mq = m_i q, \quad \text{if } q \in C_i^n, \qquad i \in \{1, \cdots, k\};$$
$$= p, \qquad \text{if } q \in X^n - \bigcup_{i=1}^{k} C_i^n.$$

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