SHORTER NOTE

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ORIENTABILITY OF HYPERSURFACES IN $\boldsymbol{R}^n$

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We give an elementary proof for the fact that a hypersurface of codimension 1 in $\boldsymbol{R}^n$ is always orientable [a surface of the type we consider is, by definition, a closed, not necessarily compact, subset of $\boldsymbol{R}^n$ that near each of its points is the set of zeros of some real-valued C^∞-function with nonzero gradient]. Another way to say this: a nonorientable differentiable manifold (without boundary) of dimension $n-1$ cannot be C^∞-embedded as closed subset of $\boldsymbol{R}^n$; here "closed" is needed as the Möbius strip (without boundary) in $\boldsymbol{R}^3$ shows. This is usually proved (for compact manifolds, without any differentiability hypothesis) by invoking Alexander duality [**2**].

Suppose we had such a nonorientable hypersurface M in $\boldsymbol{R}^n$. We take "nonorientable" to mean: there exists a loop (closed curve) in M such that the normal to M, when transported around the loop in a continuous fashion, comes back with the opposite direction. By considering a point on the normal a small distance off M, moving it around the loop and then connecting along the normal from one side of M to the other, we manufacture a closed C^∞-curve γ in $\boldsymbol{R}^n$ that meets M in exactly one point, and is transversal to M at this point [the tangent to γ is not tangent to M]. γ can be contracted to a point in $\boldsymbol{R}^n$; this amounts to a C^∞-map f of the unit disk D^2 into $\boldsymbol{R}^n$ that on the boundary S^1 of D^2 yields γ. We now make f transversal to M, i.e., we bring it into "general position" relative to M, so that $f(D^2)$ is not tangent to M at any common point; this can be done without changing f on S^1, since f is already transversal there (see [**1**], [**3**], [**4**], [**5**]). [The general transversality theorem looks formidable, but is really quite simple, particularly in the case of codimension 1; one modifies f, locally, by adding suitable affine-linear functions, multiplied by cutoff functions; in the analogous piecewise linear case one shifts the images of the vertices of some triangulation of D^2 into general position.]

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By the main property of transversality the inverse image $f^{-1}(M)$ in D^2 is very well behaved: it consists of a disjoint union of simple closed C^∞-curves in the interior of D^2 and C^∞-arcs that meet S^1 exactly in their end points (NB, each arc ends in two distinct points). But $f^{-1}(M)$ has just one point on S^1; thus the whole thing is impossible—M cannot be nonorientable.

The same proof works in the piecewise linear case and with $\boldsymbol{R}^n$ replaced by any simply connected manifold. It generalizes to prove, e.g., the well-known fact that two closed submanifolds of $\boldsymbol{R}^n$ (one of them compact) of complementary dimensions cannot meet (transversally) in an odd number of points.

References

1. R. Abraham and J. Robbin, *Transversal mappings and flows*, Benjamin, New York, 1967, p. 48.

2. P. Alexandroff and H. Hopf, *Topologie*, Springer, Berlin, 1935, p. 440.

3. H. J. Levine, *Singularities of differentiable mappings*, Mathematisches Institut der Universität Bonn, Bonn, 1959, p. 18.

4. J. Milnor, *Differential topology*, Lecture Notes, Princeton University, Princeton, N. J., 1959, Theorem 1.35, p. 22.

5. R. Thom, *Quelques propriétés globales des variétés differentiables*, Comment. Math. Helv. **28** (1954), 17–86, Theorem I.5.

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