ON LOWER BOUNDS FOR PERMANENTS OF (0, 1) MATRICES[1]

HENRYK MINC

1. **Introduction.** The permanent of an n-square matrix $A=(a_{ij})$ is defined by

$$\operatorname{per}(A) = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

If A is a (0, 1) matrix, i.e. a matrix all of whose entries are either 0 or 1, we can interpret A as the incidence matrix of a configuration of n subsets of a set consisting of n elements. In this interpretation per(A) is the number of systems of distinct representatives of the configuration. Bounds for this number are therefore of considerable combinatorial interest.

If $A=(a_{ij})$ is an n-square (0, 1) matrix, then clearly

$$0 \leqq \operatorname{per}(A) \leqq \prod_{i=1}^{n} r_i, \tag{1}$$

where $r_i=\sum_{j=1}^{n} a_{ij}$, $i=1, \cdots, n$. Several upper bounds, significantly improving the upper bound in (1), have been obtained in the last few years. On the other hand, nontrivial lower bounds for permanents of (0, 1) matrices in terms of row sums, column sums, or some other simple functions of the matrix, are difficult to establish. Indeed the permanent of an n-square (0, 1) matrix may be zero although all its row sums are $n-1$.

The first result improving the lower bound in (1) was obtained by P. Hall [3]. In the context of n-square (0, 1) matrices, Hall's theorem states that $\operatorname{per}(A)>0$ if and only if every $k\times n$ submatrix of A, $k=1, \cdots, n$, contains at least k nonzero columns.

M. Hall [2] improved the above result and showed that if A is an n-square (0, 1) matrix with a positive permanent and if $r_i \geqq t$, $i=1, \cdots, n$, for some positive integer t, then

$$\operatorname{per}(A) \geqq t!. \tag{2}$$

The inequality (2) does not provide a good lower bound if n is substantially greater than t. In fact, it is known [8] that if every row

Received by the editors September 25, 1968.

[1] Research supported by the Air Force Office of Scientific Research under grant AFOSR 698-67.

sum and column sum of A is t and $t \geqq 3$, then $\text{per}(A) \geqq n$. (See also Theorem 2 below.)

W. Jurkat and H. J. Ryser [1] obtained the following bound

$$\text{per}(A) \geqq \prod_{i=1}^{n} \max(0, r_i + 1 - i). \tag{3}$$

A simple proof of (3) was given in [5] where the case of equality was also discussed. Unfortunately the lower bound in (3) is often equal to 0, the trivial lower bound.

A new approach to the problem was recently provided by Sinkhorn [8] who proved essentially the following result. Let Λ_n^k be the set of n-square (0, 1) matrices with exactly k positive entries in each row and column. Let $k = 3m + r$ where m and r are integers, $0 \leqq r \leqq 2$. Then

$$\text{per}(A) \geqq mn + r. \tag{4}$$

In the present paper I obtain a positive lower bound for permanents of totally indecomposable n-square (0, 1) matrices:

$$\text{per}(A) \geqq \left(\sum_{i,j} a_{ij} \right) - 2n + 2. \tag{5}$$

From (5) I deduce the inequality for matrices in Λ_n^k,

$$\text{per}(A) \geqq n(k - 2) + 2 \tag{6}$$

which is better than the bound in (4).

2. **Results.** A nonnegative n-square matrix (i.e., a matrix all of whose entries are nonnegative) is called *partly decomposable* if it contains an $s \times (n-s)$ zero submatrix, $1 \leqq s \leqq n-1$. Otherwise, it is said to be *fully indecomposable.* The following characterization of fully indecomposable matrices is often useful.

LEMMA 1. *A nonnegative n-square matrix A, $n \geqq 2$, is fully indecomposable if and only if every $(n-1)$-square submatrix of A has a positive permanent.*

PROOF. The permanent of an $(n-1)$-square nonnegative matrix is zero if and only if every diagonal of the matrix contains a zero, and hence, by the Frobenius-König theorem, if and only if the matrix contains an $s \times (n-s)$ zero submatrix. The lemma follows.

COROLLARY 1. *If A is a fully indecomposable matrix and $a_{hk} > 0$, then*

$$\text{per}(A) > \text{per}(A - E_{hk}) \tag{7}$$

(*where* E_{hk} *denotes the* $n\times n$ *matrix with* 1 *as its* (h, k) *entry and zeros elsewhere*).

PROOF. Let $A(h|k)$ denote the submatrix obtained from A by deleting the hth row and the kth column of A. Expanding the permanent of A by its hth row, we have

$$\begin{aligned}\operatorname{per}(A) &= \operatorname{per}(A - E_{hk}) + a_{hk}\operatorname{per}(A(h\mid k))\\ &> \operatorname{per}(A - E_{hk}).\end{aligned}$$

The inequality is strict since A is fully indecomposable and thus, by Lemma 1, $\operatorname{per}(A(h|k))>0$.

COROLLARY 2. *If* A *is a fully indecomposable* (0, 1) *matrix and* $a_{hk}=0$, *then*

$$\operatorname{per}(A + E_{hk}) \geqq \operatorname{per}(A) + 1. \tag{8}$$

PROOF. We have

$$\begin{aligned}\operatorname{per}(A + E_{nk}) &= \operatorname{per}(A) + \operatorname{per}(A(h\mid k))\\ &\geqq \operatorname{per}(A) + 1,\end{aligned}$$

since by Lemma 1, per $(A(h|k))$ is positive and $A(h|k)$ is a (0, 1) matrix.

A nonnegative matrix is called *nearly decomposable* if it is fully indecomposable and if it has the property that the replacing of any positive entry by 0 results in a partly decomposable matrix.

Sinkhorn and Knopp [7] obtained the following remarkable result on the structure of nearly decomposable matrices.

LEMMA 2. *If* A *is a nonnegative* n*-square matrix*, $n>1$, *then there exist permutation matrices* P *and* Q *such that*

$$PAQ = \begin{bmatrix} A_1 & 0 & 0 & \cdots 0 & 0 & E_1\\ E_2 & A_2 & 0 & \cdots 0 & 0 & 0\\ 0 & E_3 & A_3 & \cdots 0 & 0 & 0\\ \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots E_{s-1} & A_{s-1} & 0\\ 0 & 0 & 0 & \cdots 0 & E_s & A_s \end{bmatrix}, \tag{9}$$

where $s\geqq 2$, *each* A_i *is a nearly decomposable square matrix and each* E_i *has exactly one positive entry.*

For proof see [7].

Before we proceed to our main theorem, we prove the following simple auxiliary result.

LEMMA 3. *Let* $n_1, \cdots, n_s$ and $\sigma_1, \cdots, \sigma_s$, $s \geqq 2$, *be positive integers satisfying* $\sigma_i = n_i = 1$ *or* $\sigma_i \geqq 2n_i$, *for each* i, $i = 1, \cdots, s$. *Then*

$$\prod_{i=1}^{s} (\sigma_i - 2n_i + 2) + 1 \geqq \sum_{i=1}^{s} (\sigma_i - 2n_i + 2) - s + 2.$$

PROOF. Use induction on s. If $\sigma_i - 2n_i + 2 \geqq 2$ for all i, the result is quite trivial, since in this case

$$\prod_{i=1}^{s} (\sigma_i - 2n_i + 2) + 1 \geqq \sum_{i=1}^{s} (\sigma_i - 2n_i + 2) + 1$$

$$> \sum_{i=1}^{s} (\sigma_i - 2n_i + 2) - s + 2.$$

Suppose then that $\sigma_i - 2n_i + 2 = 1$ for some i. We can assume without loss of generality that $\sigma_s - 2n_s + 2 = 1$. If $s = 2$ we have

$$(\sigma_1 - 2n_1 + 2)(\sigma_2 - 2n_2 + 2) + 1 = (\sigma_1 - 2n_1 + 2) + 1$$

$$= (\sigma_1 - 2n_1 + 2) + (\sigma_2 - 2n_2 + 2) - 2 + 2.$$

Next, we assume that for $s \geqq 3$,

$$\prod_{i=1}^{s-1} (\sigma_i - 2n_i + 2) + 1 \geqq \sum_{i=1}^{s-1} (\sigma_i - 2n_i + 2) - (s - 1) + 2$$

and that $\sigma_s - 2n_s + 2 = 1$. It follows that

$$\prod_{i=1}^{s} (\sigma_i - 2n_i + 2) + 1 = \prod_{i=1}^{s-1} (\sigma_i - 2n_i + 2) + 1$$

$$\geqq \sum_{i=1}^{s-1} (\sigma_i - 2n_i + 2) - (s - 1) + 2$$

$$= \sum_{i=1}^{s} (\sigma_i - 2n_i + 2) - s + 2.$$

The lemma is proved.

We now use Lemma 2 to obtain lower bound (5) for permanents of fully indecomposable (0, 1) matrices. Let $\sigma(X)$ denote the sum of all entries of matrix X.

THEOREM 1. *If* $A = (a_{ij})$ *is a fully indecomposable n-square* (0, 1) *matrix, then*

$$\operatorname{per}(A) \geqq \sigma(A) - 2n + 2. \tag{10}$$

Proof. First suppose that A is nearly decomposable. Then, by Lemma 2, there exist permutation matrices P and Q such that PAQ is of form (9). We use induction on n. If $n=1$ or 2, then (10) is actually an equality. Assume that (10) holds for all nearly decomposable t-square matrices with $t<n$. Let A_i be $n_i \times n_i$, $i=1, \cdots, s$. Then, by Corollary 2, the induction hypothesis, and Lemma 3, we have

$$\begin{aligned}
\operatorname{per}(A) = \operatorname{per}(PAQ) &\geqq \prod_{i=1}^{s} \operatorname{per}(A_i) + 1 \\
&\geqq \prod_{i=1}^{s} (\sigma(A_i) - 2n_i + 2) + 1 \\
&\geqq \sum_{i=1}^{s} (\sigma(A_i) - 2n_i + 2) - s + 2 \\
&= \sum_{i=1}^{s} \sigma(A)_i + s - 2\sum_{i=1}^{s} n_i + 2 = \sigma(A) - 2n + 2.
\end{aligned}$$

This proves (10) in case A is nearly decomposable.

Now suppose that A is any fully indecomposable matrix. If A is not nearly decomposable, there must exist a positive entry in A, $a_{i_1j_1}=1$, such that $A-E_{i_1j_1}$ is a fully indecomposable (0, 1) matrix. If $A-E_{i_1j_1}$ is not nearly decomposable, then there exists a positive entry in $A-E_{i_1j_1}$, $a_{i_2j_2}=1$, such that $A-E_{i_1j_1}-E_{i_2j_2}$ is fully indecomposable, and so on. Thus we must finally obtain a nearly decomposable matrix B satisfying

$$A = B + \sum_{t=1}^{m} E_{i_tj_t}.$$

We can now use Corollary 2, m times, to conclude that

$$\operatorname{per}(A) \geqq \operatorname{per}(B) + m,$$

and hence applying inequality (10) to nearly decomposable matrix B we have

$$\operatorname{per}(A) \geqq \sigma(B) - 2n + 2 + m.$$

But $\sigma(B)+m=\sigma(A)$ and thus

$$\operatorname{per}(A) \geqq \sigma(A) - 2n + 2.$$

This concludes the proof of Theorem 1.

The bound in Theorem 1 cannot be usefully applied to the case of decomposable (0, 1) matrices. However, if A belongs to a more restricted class of (0, 1) matrices then a significant lower bound can be deduced from Theorem 1, even when A is decomposable.

Let Λ_n^k denote the set of n-square (0, 1) matrices each with exactly k ones in each row and column. In a recent paper [8] Sinkhorn proved that if $A \in \Lambda_n^3$ then

$$\text{per}(A) \geqq n, \tag{11}$$

and he concluded that

$$\lim_{n\to\infty} \left(\operatorname*{Inf}_{A\in\Lambda_n^3} \text{per}(A) \right) = +\infty.$$

This answered in the affirmative a conjecture of M. Hall. In our next theorem we improve inequality (11).

THEOREM 2. *If* $A \in \Lambda_n^k$ *then*

$$\text{per}(A) \geqq n(k-2)+2. \tag{12}$$

PROOF. If A is fully indecomposable then (12) holds by virtue of Theorem 1 since $\sigma(A) = nk$. If A is partly decomposable, then there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix},$$

where A_i is n_i-square, $i = 1, 2$. Since each row sum in A_1 is k, it follows that $\sigma(A_1) = n_1 k$ and thus all the positive entries in the first n_1 columns of PAQ are in A_i, and A_3 must be 0. Hence

$$PAQ = A_1 \dotplus A_2$$

where $A_1 \in \Lambda_{n_1}^k$ and $A_2 \in \Lambda_{n_2}^k$. If $k=1$, then the theorem is trivial. Assume therefore that $k \geqq 2$ and use induction on n. We have

$$\begin{aligned} \text{per}(A) &= \text{per}(A_1)\,\text{per}(A_2) \\ &\geqq (n_1(k-2)+2)(n_2(k-2)+2) \\ &> (n_1+n_2)(k-2)+2 = n(k-2)+2. \end{aligned}$$

This concludes the proof of Theorem 2.

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University of California, Santa Barbara