GROUPS WHICH ARE COGROUPS

M. L. CURTIS AND J. DUGUNDJI

It is the purpose of this note to give an elementary proof that $S^0$, $S^1$, $S^3$ are the only spheres which can be made into topological groups. This was proved by Samelson in [3]. He showed that a compact Lie group which is a homotopy sphere must have rank 1, and then that a compact Lie group of rank 1 has dimension 1 or 3. We get the first part easily by showing that a compact Lie group which is an $H$-cogroup (e.g., a suspension) must have rank 1. The second part closely follows Samelson's proof. For basic facts about $H$-groups and $H$-cogroups the reader is referred to §§5 and 6 of Chapter I of Spanier [5].

If $G$ is an $H$-group which is also an $H$-cogroup, then the set $[G, G]_0$ of pointed homotopy classes is an abelian group with its operation given equally by the $H$-cogroup structure on the domain $G$ or the $H$-group structure on the range $G$ [5, page 44]. From this the following lemma is immediate.

**Lemma 1.** If $G$ is an $H$-group which is also an $H$-cogroup, $i: G \rightarrow G$ is the identity map and $\psi: G \rightarrow G$ is the squaring map $x \mapsto x^2$, then

$$[\psi] = 2[i] \in [G, G]_0.$$ 

Next we need a special case of a result due to Hopf [2].

**Lemma 2.** If $G$ is a compact connected Lie group of rank $r$ and $\psi$ is the squaring map, then degree $\psi = 2^r$.

**Proof.** We will count the algebraic number of times a point $y \in G$ is covered. We may assume $y$ is regular (i.e., lies in a unique maximal torus) because the set of singular (not regular) points forms a set of dimension less than the dimension of $G$ (indeed, the dimension is at most $\dim G - 3$; see [4, p. 19]).

Let $T$ be the unique maximal torus containing $y$. Since $\dim T = r$ we easily find that $y$ has exactly $2^r$ square roots in $T$. If $z$ is any other square root of $y$, then $z$ lies in some other maximal torus $T' \neq T$ (since maximal tori cover $G$). But then $y = z^2 \in T'$, contradicting the regularity of $y$. Thus there are exactly $2^r$ points $x \in G$ with $\psi(x) = y$.

We assert that all such $x$ cover $y$ with the same orientation. Given $x$ with $x^2 = y$, let $U$ be a small open neighborhood of $x$ containing no other roots of $y$ and such that $U$ is star-shaped relative to $x$. Then

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$\psi: U \to G$

is a diffeomorphism (if $U$ is small enough). Furthermore, if $T_x: U \to G$ is translation by $x$ it is easy to get an isotopy between $\psi$ and $T_x$ on $U$. Since $T_x$ preserves orientation, so does $\psi$. Hence a neighborhood of $y$ is covered $2^r$ times with the same orientation. Hence the degree of $\psi$ is $2^r$.

From Lemmas 1 and 2 we get

**Theorem 1.** If a compact Lie group $G$ is also an $H$-cogroup, then $\text{rank } G = 1$.

**Corollary.** If $n \geq 1$ and $S^n$ can be made into a topological group, then $S^n$ is a Lie group of rank 1.

**Proof.** A group manifold is a Lie group, and $S^n$ is compact, so Theorem 1 applies.

**Theorem 2 (following Samelson).** If $G$ is a compact Lie group of rank 1, and $\dim G = n > 1$, then $n = 3$.

**Proof.** Since $G$ is compact we can get an inner product on its Lie algebra $\mathfrak{g}$ such that the adjoint representation

$$\text{Ad}: G \to GL(\mathfrak{g})$$

is such that each $\text{Ad}(g)$ is orthogonal; i.e., $\text{Ad}(G) \subset O(\mathfrak{g})$ the orthogonal group on $\mathfrak{g}$.

Let $T = S^1$ be a maximal torus of $G$ and let $X$ be a unit vector in $\mathfrak{g}$ tangent to $T$. Define

$$\phi': G \to S^{n-1} \subset \mathfrak{g}$$

by $\phi'(g) = \text{Ad } gX$.

**Lemma.** $\phi'$ induces a homeomorphism $\phi: G/T \to S^{n-1}$.

**Proof of Lemma.** First off, $\phi'$ does induce $\phi$ since if $t \in T$

$$\phi'(gt) = \text{Ad}(gt)X = \text{Ad}(g)\text{Ad}(t)X = \text{Ad}(g)X = \phi'(g).$$

Second, $\phi$ is injective for if

$$\text{Ad } gX = \text{Ad } hX,$$

then $\text{Ad}(gh^{-1})X = X$. Thus $\text{Ad}(gh^{-1})$ leaves the one-dimensional Lie algebra $\mathfrak{L}(T)$ of $T$ elementwise fixed. Since $T$ is maximal this implies $gh^{-1} \in T$.

Finally, $G/T$ is a compact closed $n - 1$ manifold and it follows that $\phi$ is surjective, so the Lemma is proved.
Since \( \phi \) maps \( G/T \) onto \( S^{n-1} \) it follows that there exists \( g_0 \in G \) such that \( \text{Ad} \ g_0 \ X = -X \). This allows us to get a map \( f : S^2 \to S^{n-1} \) as follows. Let \( \omega : [0, 1] \to G \) be a path from the identity \( e \) to \( g_0 \). Then since \( T = S^1 \) we have a map

\[
F : S^1 \times I \to G
\]
given by \( F(t, \tau) = \omega(t) \omega(t)^{-1} \). Then \( F(S^1 \times 0) \subseteq S^1 \) and \( F(S^1 \times 1) \subseteq S^1 \) so \( F \) induces a map \( f : S^2 \to G/S^1 = S^{n-1} \). We assert that \( f \) is essential.

If \( f \) is nullhomotopic, we can cover a nullhomotopy in \( G \) to get a homotopy

\[
\Phi : T \times I \to T
\]
with \( \Phi_0 = F(-, 0) \) the identity and \( \Phi_1 = F(1, 1) \) sends \( t \) to \( t^{-1} \). No such homotopy exists, so \( f \) is essential and we conclude that \( S^{n-1} = S^2 \) and \( n = 3 \).

**Theorem 3.** The \( n \)-sphere \( S^n \) (\( n > 0 \)) can be made into a topological group if and only if \( n = 1 \) or \( n = 3 \).

**Proof.** Use Theorem 2 and the corollary to Theorem 1.

**Remark.** \( SO(3) \) is a compact Lie group of rank 1 and dimension 3, so that it could be a cogroup as far as the results above are concerned. But Hilton [1] has shown that a comultiplication on \( X \) implies \( X \) has Lusternik-Schnirelmann category \( \leq 2 \), whereas \( SO(3) = P^3 \) has category 4.

**References**