GROUPS WHICH ARE COGROUPS

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It is the purpose of this note to give an elementary proof that S^0, S^1, S^3 are the only spheres which can be made into topological groups. This was proved by Samelson in [3]. He showed that a compact Lie group which is a homotopy sphere must have rank 1, and then that a compact Lie group of rank 1 has dimension 1 or 3. We get the first part easily by showing that a compact Lie group which is an H-cogroup (e.g., a suspension) must have rank 1. The second part closely follows Samelson's proof. For basic facts about H-groups and H-cogroups the reader is referred to §§5 and 6 of Chapter I of Spanier [5].

If G is an H-group which is also an H-cogroup, then the set $[G, G]_0$ of pointed homotopy classes is an abelian group with its operation given equally by the H-cogroup structure on the domain G or the H-group structure on the range G [5, page 44]. From this the following lemma is immediate.

LEMMA 1. *If G is an H-group which is also an H-cogroup, $i: G \to G$ is the identity map and $\psi: G \to G$ is the squaring map $x \mapsto x^2$, then*

$$[\psi] = 2[i] \in [G, G]_0.$$

Next we need a special case of a result due to Hopf [2].

LEMMA 2. *If G is a compact connected Lie group of rank r and ψ is the squaring map, then degree $\psi = 2^r$.*

PROOF. We will count the algebraic number of times a point $y \in G$ is covered. We may assume y is regular (i.e., lies in a unique maximal torus) because the set of singular (=not regular) points forms a set of dimension less than the dimension of G (indeed, the dimension is at most dim $G-3$; see [4, p. 19]).

Let T be the unique maximal torus containing y. Since dim $T = r$ we easily find that y has exactly 2^r square roots in T. If z is any other square root of y, then z lies in some other maximal torus $T' \neq T$ (since maximal tori cover G). But then $y = z^2 \in T'$, contradicting the regularity of y. Thus there are exactly 2^r points $x \in G$ with $\psi(x) = y$.

We assert that all such x cover y with the same orientation. Given x with $x^2 = y$, let U be a small open neighborhood of x containing no other roots of y and such that U is star-shaped relative to x. Then

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$$\psi: U \to G$$

is a diffeomorphism (if U is small enough). Furthermore, if T_x: $U \to G$ is translation by x it is easy to get an *isotopy* between ψ and T_x on U. Since T_x preserves orientation, so does ψ. Hence a neighborhood of y is covered 2^r times with the same orientation. Hence the degree of ψ is 2^r.

From Lemmas 1 and 2 we get

THEOREM 1. *If a compact Lie group G is also an H-cogroup, then rank $G=1$.*

COROLLARY. *If $n \geqq 1$ and S^n can be made into a topological group, then S^n is a Lie group of rank* 1.

PROOF. A group manifold is a Lie group, and S^n is compact, so Theorem 1 applies.

THEOREM 2 (FOLLOWING SAMELSON). *If G is a compact Lie group of rank* 1, *and* dim $G=n>1$, *then $n=3$.*

PROOF. Since G is compact we can get an inner product on its Lie algebra $\mathfrak{g}$ such that the adjoint representation

$$\mathrm{Ad}: G \to GL(\mathfrak{g})$$

is such that each $\mathrm{Ad}(g)$ is orthogonal; i.e., $\mathrm{Ad}(G) \subset \mathcal{O}(\mathfrak{g})$ the orthogonal group on $\mathfrak{g}$.

Let $T=S^1$ be a maximal torus of G and let X be a unit vector in $\mathfrak{g}$ tangent to T. Define

$$\phi': G \to S^{n-1} \subset \mathfrak{g}$$

by $\phi'(g) = \mathrm{Ad}\ gX$.

LEMMA. *ϕ' induces a homeomorphism $\phi: G/T \to S^{n-1}$.*

PROOF OF LEMMA. First off, ϕ' does induce ϕ since if $t \in T$

$$\phi'(gt) = \mathrm{Ad}(gt)X = \mathrm{Ad}(g)\mathrm{Ad}(t)X = \mathrm{Ad}(g)X = \phi'(g).$$

Second, ϕ is injective for if

$$\mathrm{Ad}\ gX = \mathrm{Ad}\ hX,$$

then $\mathrm{Ad}(gh^{-1})X = X$. Thus $\mathrm{Ad}(gh^{-1})$ leaves the one-dimensional Lie algebra $\mathfrak{L}(T)$ of T elementwise fixed. Since T is maximal this implies $gh^{-1} \in T$.

Finally, G/T is a compact closed $n-1$ manifold and it follows that ϕ is surjective, so the Lemma is proved.

Since ϕ maps G/T onto S^{n-1} it follows that there exists $g_0 \in G$ such that $\mathrm{Ad}\, g_0 X = -X$. This allows us to get a map $f: S^2 \to S^{n-1}$ as follows. Let $\omega: [0, 1] \to G$ be a path from the identity e to g_0. Then since $T = S^1$ we have a map

$$F: S^1 \times I \to G$$

given by $F(t, \tau) = \omega(\tau) t \omega(\tau)^{-1}$. Then $F(S^1 \times 0) \subset S^1$ and $F(S^1 \times 1) \subset S^1$ so F induces a map $f: S^2 \to G/S^1 = S^{n-1}$. We assert that f is essential.

If f is nullhomotopic, we can cover a nullhomotopy in G to get a homotopy

$$\Phi: T \times I \to T$$

with $\Phi_0 = F(-, 0)$ the identity and $\Phi_1 = F(1, 1)$ sends t to t^{-1}. No such homotopy exists, so f is essential and we conclude that $S^{n-1} = S^2$ and $n = 3$.

THEOREM 3. *The n-sphere S^n $(n > 0)$ can be made into a topological group if and only if $n = 1$ or $n = 3$.*

PROOF. Use Theorem 2 and the corollary to Theorem 1.

REMARK. SO(3) is a compact Lie group of rank 1 and dimension 3, so that it could be a cogroup as far as the results above are concerned. But Hilton [1] has shown that a comultiplication on X implies X has Lusternik-Schnirelmann category $\leqq 2$, whereas $\mathrm{SO}(3) = P^3$ has category 4.

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