A NOTE ON GENERATING SETS FOR INVERTIBLE IDEALS[1]

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It is well known that an invertible fractional ideal of a commutative ring with identity must be finitely generated. S. U. Chase has shown that for any positive integer n, there is an integral domain W_n containing an invertible ideal with a basis of n, but no fewer, generators. Chase's example does not appear in the literature, but the example has been referred to by Bass in [**1**, 541], by Swan in [**4**, 270], and by Gilmer and Heinzer in [**2**]. We know of no verification of the details of Chase's example independent of Swan's results in [**4**]; these results of Swan are quite involved.

In this note, we give an example of a domain D_n with identity containing an invertible ideal A_n with a basis of n, but no fewer, generators. The domain D_n is related to, but not isomorphic to, Chase's domain W_n. However, our verification that A_n has no basis of fewer than n elements depends only upon Lemma 1, a result which should be of independent interest in itself. The proof of Lemma 1 requires essentially only the Borsuk-Ulam Theorem [**3**, 152].

We use E to denote the field of real numbers; n is a positive integer greater than one, and $\{X_i\}_{i=1}^n$ is a set of indeterminates over E.

LEMMA 1. *If $f_1, \cdots, f_{n-1} \in E[X_1, \cdots, X_n]$, where each nonzero monomial of each f_i has odd degree, and if*

$$S^n = \left\{ u = (u_1, \cdots, u_n) \in E^n \,\middle|\, \sum_{i=1}^{n} u_i^2 = \alpha \right\},$$

where α is a fixed positive real number, then $f_1, \cdots, f_{n-1}$ have a common zero on S^n.

PROOF. Consider the mapping $g: S^n \to E^{n-1}$ defined by $g(u) = (f_1(u), \cdots, f_{n-1}(u))$. g is continuous, and the hypothesis on the f_i's implies that $g(u) = -g(-u)$ for any $u \in S^n$. By the Borsuk-Ulam Theorem, there exists a pair $\{u, -u\}$ of antipodal points of S^n such that $g(u) = g(-u)$. Consequently, $g(u) = (0, \cdots, 0)$, and u is a common zero of the f_i's on S^n.

Now let $J_n = E[\{X_iX_j\}_{1 \le i,j \le n}]$, let $s = \sum_{i=1}^n X_i^2$, and let $D_n = (J_n)_N$, where $N = \{s^i\}_{i=0}^{\infty}$. If A_n is the ideal of D_n generated by $\{X_1X_i\}_{1 \le i \le n}$,

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then A_n is invertible, for clearly $A_n^2 \subseteq X_1^2 D_n$, but equality holds since $X_1^2 = X_1^2 s/s = \sum_{i=1}^n (X_1 X_i)^2/s \in A_n^2$. Therefore, A_n is invertible and has a basis of n elements. But if $g_1, \cdots, g_{n-1} \in A_n$, then there exist polynomials $f_1, \cdots, f_{n-1} \in E[X_1, \cdots, X_n]$ such that each nonzero monomial in each f_i has odd degree and such that $g_i = X_1 f_i$ for each i. By Lemma 1 the f_i's, and hence the g_i's, have a common zero $u = (u_1, \cdots, u_n)$ in E^n other than the origin. The equality $A_n = \{g_1, \cdots, g_{n-1}\} D_n$ would imply the existence of a subset $\{h_{ij}\}_{1 \leq i, j \leq n-1}$ of J_n and a positive integer r such that

$$X_1 X_k s^r = \sum_{j=1}^{n-1} h_{kj} X_1 f_j$$

for $1 \leqq k \leqq n$. Thus

$$X_k s^r = \sum_{j=1}^{n-1} h_{kj} f_j$$

and

$$u_k [s(u)]^r = \sum_{j=1}^{n-1} h_{kj}(u) f_j(u) = 0.$$

Since $s(u) \neq 0$, it then follows that $u_k = 0$ for each k, a contradiction. Hence A_n has no basis of $n-1$ elements.

References

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