EXPANSION OF ANALYTIC FUNCTIONS IN EXPONENTIAL POLYNOMIALS

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The expansion of analytic functions in interpolation series by the method of kernel expansion has been treated in [3], [4] and the detailed application of the method to expansions in the classical polynomials has culminated in the well-known monograph of Boas and Buck [2]. In the present paper we use the kernel expansion technique to obtain a new expansion in which the terms are not polynomials in the usual sense of powers but instead are the exponential polynomials

\[ e_n(z) = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (n + 1 - k)^{z}. \]

These exponential polynomials together with the linear functionals

\[ T_n(f) = \binom{\Delta}{n} f(0) = \Delta(\Delta - 1) \cdots (\Delta - n + 1) f(0)/n! \]

form a biorthogonal system \((T_n(e_m) = 0 \text{ if } n \neq m, T_n(e_n) = 1)\), giving the formal interpolation series

\[ f(z) \sim \sum_{n=0}^{\infty} \binom{\Delta}{n} f(0) e_n(z). \]

Our main result will be the establishment of precise conditions under which (1) actually converges to \(f(z)\).

The chief analytical device of the method of kernel expansion is the Pólya representation of analytic functions which we may state as follows: If \(f(z) = \sum_{n=0}^{\infty} a_n z^n/n!\) is entire and of exponential type, then

\[ f(z) = (2\pi i)^{-1} \int_{\Gamma} e^{zw} F(w) dw, \]

where \(F(w) = \sum_{n=0}^{\infty} a_n w^{n+1}\) and \(\Gamma\) encircles the set \(D(f)\) consisting of the singular points of \(F(w)\) and the points exterior to the domain of \(F\). It is well known that \(D(f)\) is related to the growth of \(f(z)\). (See [1, Chapter 5].)

The representation (2) may also be used to represent many of the most-used linear functionals of analysis in the form

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and several instances will be seen below. The function \( g_n(w) \) is called the *generator* of \( T_n \).

Now if the kernel expansion
\[
e^{zw} = \sum_0^\infty u_n(z) g_n(w)
\]
holds uniformly for all \( w \) on a simple contour \( \Gamma \) which encircles \( D(f) \) and for all \( z \), we may integrate termwise in (2) and obtain
\[
f(z) = \sum_0^\infty T_n(f) u_n(z), \text{ for all } z.
\]

The method just outlined has been applied by Buck [4] to the classical Newton series. In that case one has the orthonormal system
\[
u_n(z) = \binom{z-1}{n}, \quad T_n(f) = \Delta^nf(1),
\]
yielding the formal expansion
\[
f(z) \sim \sum_0^\infty \Delta^nf(1) \binom{z-1}{n}.
\]
The representation (3) for \( T_n \) now can be seen to require \( g_n(w) = e^w(e^w - 1)^n \). The kernel expansion (4) takes the form
\[
e^{zw} = e^w [1 + (e^w - 1)]^{z-1} = e^w \sum_0^\infty \binom{z-1}{n} (e^w - 1)^n
\]
valid in the region defined by \( |e^w - 1| < 1 \). Termwise integration yields the following result. If \( f(z) \) is an entire function of exponential type such that \( D(f) \) is contained in \( |e^w - 1| < 1 \), then \( f(z) \) admits the convergent Newton series expansion (5) for all \( z \).

The result just given is already stronger than a classical result in Whittaker [8] and still stronger forms are given in [2]. However no result yet obtained by kernel expansion is as strong as that given in Nörlund [7], which we require below and state herewith in somewhat less generality than given in Nörlund.

**Theorem (Nörlund).** If \( f(z) \) is analytic and holomorphic in the half-plane \( R(z) \geq \alpha \) and satisfies in that half-plane the inequality
\[
|f(\alpha + re^{i\theta})| < e^\log 2(1 + r)^{\beta + \epsilon(r)}, \quad -\pi/2 \leq \theta \leq \pi/2,
\]
where \( \epsilon(r) \rightarrow 0 \) as \( r \rightarrow \infty \), then \( f(z) \) admits the convergent expansion (5) for all \( z \) satisfying \( R(z) \geq \max(\alpha, \beta + \frac{\pi}{2}) \). Furthermore the convergence is uniform in any bounded subset of the indicated right half-plane of convergence.
The function \( f(z) = z^\gamma \), \( \gamma \) any complex number, can be shown (see [5]) to satisfy the conditions required in the previous theorem with \( \alpha \geq \delta > 0, \beta = -\frac{1}{2} \). Thus we have

\[
(6) \quad z^\gamma = \sum_{n=0}^{\infty} \Delta^n z^\gamma \bigg|_{n=1} \left( \frac{n!}{n} \right),
\]

uniformly in any bounded region of the half-plane \( R(z) \geq \epsilon > 0 \), for arbitrary \( \gamma \). It is understood that the branch of \( z^\gamma \) be chosen consistently throughout (6) so for definiteness we choose the branch satisfying \(-\pi/2 < \arg z < \pi/2\).

An expansion of the form (4) is obtained if we set \( z = e^{\omega}, \gamma = z \) in (6). We have

\[
(7) \quad e^{\omega z} = \sum_{n=0}^{\infty} \Delta^n e^{\omega z} \bigg|_{n=1} \left( e^{\omega z} - 1 \right).
\]

Uniform convergence is obtained in any bounded region in the strip \( |I(w)| \leq \delta < \pi/2 \). This choice of strip is also consistent with the branch specification above.

The generators \( g_n(w) \) are now

\[
\left( e^{\omega z} - 1 \right)
\]

and by virtue of the representation (3) the functionals are found to be

\[
T_n(f) = \binom{\Delta}{n} f(0).
\]

We integrate termwise around a simple contour enclosing \( D(f) \) and contained in \( |I(w)| \leq \delta < \pi/2 \). Using the fact that \( D(f) \) is closed and bounded for any entire function of exponential type, we obtain our main result, as follows:

**Theorem.** Any entire function of exponential type such that \( D(f) \) lies in the strip \( |I(w)| < \pi/2 \) admits the convergent exponential interpolation series expansion

\[
f(z) = \sum_{n=0}^{\infty} \binom{\Delta}{n} f(0)e_n(z),
\]

for all \( z \), where \( e_n(z) \) is the exponential polynomial

\[
e_n(z) = \Delta^n x^n \bigg|_{n=1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n + 1 - k)^z.
\]
REMARK. It has been pointed out by the referee that Macintyre [6] has given a refinement of the theorem of Nörlund stated above, establishing convergence in $R(z) > \max(\alpha, \beta)$.

BIBLIOGRAPHY


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