EXPANSION OF ANALYTIC FUNCTIONS IN EXPONENTIAL POLYNOMIALS

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The expansion of analytic functions in interpolation series by the method of kernel expansion has been treated in [3], [4] and the detailed application of the method to expansions in the classical polynomials has culminated in the well-known monograph of Boas and Buck [2]. In the present paper we use the kernel expansion technique to obtain a new expansion in which the terms are not polynomials in the usual sense of powers but instead are the exponential polynomials

$$e_n(z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+1-k)^z.$$

These exponential polynomials together with the linear functionals

$$T_n(f) = \binom{\Delta}{n} f(0) = \Delta(\Delta-1)\cdots(\Delta-n+1)f(0)/n!$$

form a biorthogonal system ($T_n(e_m)=0$ if $n \neq m$, $T_n(e_n)=1$), giving the formal interpolation series

$$f(z) \sim \sum_{0}^{\infty} \binom{\Delta}{n} f(0) e_n(z). \tag{1}$$

Our main result will be the establishment of precise conditions under which (1) actually converges to $f(z)$.

The chief analytical device of the method of kernel expansion is the Pólya representation of analytic functions which we may state as follows: If $f(z) = \sum_0^\infty a_n z^n/n!$ is entire and of exponential type, then

$$f(z) = (2\pi i)^{-1} \int_\Gamma e^{zw} F(w)\, dw, \tag{2}$$

where $F(w) = \sum_0^\infty a_n/w^{n+1}$ and Γ encircles the set $D(f)$ consisting of the singular points of $F(w)$ and the points exterior to the domain of F. It is well known that $D(f)$ is related to the growth of $f(z)$. (See [1, Chapter 5].)

The representation (2) may also be used to represent many of the most-used linear functionals of analysis in the form

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$$T_n(f) = (2\pi i)^{-1} \int_\Gamma g_n(w) F(w) dw, \tag{3}$$

and several instances will be seen below. The function $g_n(w)$ is called the *generator* of T_n.

Now if the kernel expansion

$$e^{zw} = \sum_0^\infty u_n(z) g_n(w) \tag{4}$$

holds uniformly for all w on a simple contour Γ which encircles $D(f)$ and for all z, we may integrate termwise in (2) and obtain $f(z) = \sum_0^\infty T_n(f)\, u_n(z)$, for all z.

The method just outlined has been applied by Buck [4] to the classical Newton series. In that case one has the orthonormal system

$$u_n(z) = \binom{z-1}{n}, \qquad T_n(f) = \Delta^n f(1),$$

yielding the formal expansion

$$f(z) \sim \sum_0^\infty \Delta^n f(1) \binom{z-1}{n}. \tag{5}$$

The representation (3) for T_n now can be seen to require $g_n(w) = e^w(e^w - 1)^n$. The kernel expansion (4) takes the form

$$e^{zw} = e^w[1 + (e^w - 1)]^{z-1} = e^w \sum_0^\infty \binom{z-1}{n} (e^w - 1)^n$$

valid in the region defined by $|e^w - 1| < 1$. Termwise integration yields the following result. If $f(z)$ is an entire function of exponential type such that $D(f)$ is contained in $|e^w - 1| < 1$, then $f(z)$ admits the convergent Newton series expansion (5) for all z.

The result just given is already stronger than a classical result in Whittaker [8] and still stronger forms are given in [2]. However no result yet obtained by kernel expansion is as strong as that given in Nörlund [7], which we require below and state herewith in somewhat less generality than given in Nörlund.

THEOREM (NÖRLUND). *If $f(z)$ is analytic and holomorphic in the half-plane $R(z) \geqq \alpha$ and satisfies in that half-plane the inequality $|f(\alpha + re^{i\theta})| < e^{r \log 2}(1+r)^{\beta+\epsilon(r)}$, $-\pi/2 \leqq \theta \leqq \pi/2$, where $\epsilon(r) \to 0$ as $r \to \infty$, then $f(z)$ admits the convergent expansion* (5) *for all z satisfying $R(z) > \max(\alpha, \beta + \frac{1}{2})$. Furthermore the convergence is uniform in any bounded subset of the indicated right half-plane of convergence.*

The function $f(z)=z^\gamma$, γ any complex number, can be shown (see [5]) to satisfy the conditions required in the previous theorem with $\alpha\geqq\delta>0$, $\beta=-\frac{1}{2}$. Thus we have

$$z^\gamma = \sum_0^\infty \Delta^n z^\gamma \big|_{z=1} \binom{z-1}{n}, \tag{6}$$

uniformly in any bounded region of the half-plane $R(z)\geqq\epsilon>0$, for arbitrary γ. It is understood that the branch of z^γ be chosen consistently throughout (6) so for definiteness we choose the branch satisfying $-\pi/2<\arg z<\pi/2$.

An expansion of the form (4) is obtained if we set $z=e^w$, $\gamma=z$ in (6). We have

$$e^{zw} = \sum_0^\infty \Delta^n x^z \big|_{x=1} \binom{e^w-1}{n}. \tag{7}$$

Uniform convergence is obtained in any bounded region in the strip $|I(w)|\leqq\delta<\pi/2$. This choice of strip is also consistent with the branch specification above.

The generators $g_n(w)$ are now

$$\binom{e^w-1}{n}$$

and by virtue of the representation (3) the functionals are found to be

$$T_n(f) = \binom{\Delta}{n} f(0).$$

We integrate termwise around a simple contour enclosing $D(f)$ and contained in $|I(w)|\leqq\delta<\pi/2$. Using the fact that $D(f)$ is closed and bounded for any entire function of exponential type, we obtain our main result, as follows:

THEOREM. *Any entire function of exponential type such that $D(f)$ lies in the strip $|I(w)|<\pi/2$ admits the convergent exponential interpolation series expansion*

$$f(z) = \sum_0^\infty \binom{\Delta}{n} f(0) e_n(z),$$

for all z, where $e_n(z)$ is the exponential polynomial

$$e_n(z) = \Delta^n x^z \big|_{x=1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+1-k)^z.$$

REMARK. It has been pointed out by the referee that Macintyre [6] has given a refinement of the theorem of Nörlund stated above, establishing convergence in $R(z) > \max(\alpha, \beta)$.

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