SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ON RAMANUJAN'S SUMMATION OF ${}_1\psi_1(a; b; z)$

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In [1] a very short proof of Jacobi's triple product was given. It turns out that by suitably modifying the technique utilized in [1] a proof may be given of the following generalization of Jacobi's identity due to Ramanujan [2, p. 194, Equation (1.3)].

$$(1)\qquad {}_1\psi_1(a;b;z) = \prod_{n=0}^{\infty} \frac{(1-bq^n/a)(1-azq^n)(1-q^{n+1}/az)(1-q^{n+1})}{(1-q^{n+1}/a)(1-bq^n/az)(1-bq^n)(1-zq^n)}.$$

Our proof will utilize the q-analog of Gauss's theorem [3, p. 97, Equation (3.3.2.5)] in the following form:

$$(2)\qquad \sum_{n=0}^{\infty} \frac{(a)_n(b)_n t^n}{(q)_n(abt)_n} = \prod_{n=0}^{\infty} \frac{(1-atq^n)(1-btq^n)}{(1-abtq^n)(1-tq^n)},$$

where $(a)_n = \prod_{j=0}^{n-1}(1-aq^j)$.

Thus

$$\prod_{m=0}^{\infty} \frac{(1-bq^m/a)(1-q^{m+1})(1-azq^m)}{(1-q^{m+1}/a)(1-bq^m)(1-zq^m)}$$

$$= \prod_{m=0}^{\infty} \frac{(1-bq^m/a)(1-q^{m+1})}{(1-q^{m+1}/a)(1-bq^m)} \sum_{n=-\infty}^{\infty} (a)_n z^n/(q)_n$$

(by (2) with $b = 0$, $t = z$; terms with $n < 0$ are identically zero)

$$= \sum_{n=-\infty}^{\infty} \frac{(a)_n z^n}{(b)_n} \prod_{m=0}^{\infty} \frac{(1-bq^m/a)(1-q^{n+m+1})}{(1-bq^{n+m})(1-q^{m+1}/a)}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(a)_n z^n}{(b)_n} \sum_{m=0}^{\infty} \frac{(aq^n)_m(b/q)_m(q/a)^m}{(q)_m(bq^n)_m} \qquad \text{(by (2))}$$

$$= \sum_{m=0}^{\infty} \frac{(b/q)_m(q/az)^m}{(q)_m} \sum_{n=-\infty}^{\infty} \frac{(a)_{n+m}z^{n+m}}{(b)_{n+m}} = \prod_{m=0}^{\infty} \frac{(1-bq^m/az)}{(1-q^{m+1}/az)}\, {}_1\psi_1(a;b;z),$$

where the last equation is obtained from the preceding by shifting the index of summation back by m. The above expansions and inter-

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changes of summation are justified only if $|z|<1$, $|q|<1$, $|b/az|<1$, $|q/a|<1$, $|q/az|<1$; however, once (1) is established, analytic continuation implies the result is valid provided only $|b/a|<|z|<1$, $|q|<1$.

References

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