RIESZ MATRICES THAT ARE ALSO HAUSDORFF MATRICES

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If $A = (a_0, a_1, a_2, \ldots)$ is an infinite sequence, $a_0 > 0$, $a_p \geq 0$ for $p \geq 1$ and $S$ is the sequence of partial sums of $A$ then by the Riesz matrix for $A$ we shall mean the infinite triangular matrix $[A]$ such that $[A]_{np} = a_p/S_n$, $0 \leq p \leq n$ and $[A]_{np} = 0$ otherwise.

If $D = (d_0, d_1, d_2, \ldots)$ is an infinite number sequence, $A^\infty d_p = d_p$ and, for each $n$, $\Delta^{n+1} d_p = \Delta^n d_p - \Delta^n d_{p+1}$, then the Hausdorff matrix for the sequence $D$ is the infinite triangular matrix $[H(D)]$ such that $[H(D)]_{np} = C_n \Delta^{-n} d_p$, $0 \leq p \leq n$, and $[H(D)]_{np} = 0$ otherwise.

In considering the Riesz matrix for the sequence $A$ we may, without loss of generality, consider only sequences with $a_0 = 1$. It is well known that the Riesz matrix for the sequence $A$ is regular only in case the sequence of partial sums of $A$ diverges, and that the Hausdorff matrix for the sequence $D$ is regular only in case there exists a function, $\alpha$, of bounded variation on $[0, 1]$, such that $\alpha(0+) = \alpha(0)$, $\alpha(1) - \alpha(0) = 1$ and for each nonnegative integer $p$, $d_p = \int_0^1 t^p d\alpha(t)$.

It was noticed by Garabedian and Wall [1, pp. 198–199] that a certain class of hypergeometric summability matrices were also Riesz matrices. The following theorem establishes necessary and sufficient conditions that a Riesz matrix for the sequence $A$ also be a regular Hausdorff matrix.

**Theorem.** If $A$ is a sequence such that $a_0 = 1$ and for each $p \geq 1$, $a_p \geq 0$ and $[A]$ is the Riesz matrix for $A$ then the following two statements are equivalent:

1. There is a positive number $x$ such that $A = (1, x, x(x+1)/2, \ldots, x(x+1)\cdots(x+p-1)/p!, \ldots)$.
2. The matrix $[A]$ is a regular Hausdorff matrix.

Suppose (1) is true. The sequence of partial sums of $A$ is as follows; $S_0 = 1$ and if $p \geq 1$ then $S_p = (1+x)(2+x)\cdots(p+x)/p!$. A short computation will show that if each of $n$ and $p$ is a nonnegative integer, $p \leq n$, then $[A]_{np} = C_n (x(n-p)\cdots(x+p-1)/p!/(x+p)(x+p+1)\cdots(x+n))$. In particular $[A]_{nn} = x/(x+n)$. An induction argument shows that if each of $n$ and $p$ is a nonnegative integer, $p \leq n$, then $\Delta^n [A]_{pp} = (n!)x/(x+p)(x+p+1)\cdots(x+p+n)$. In particular $C_n \Delta^{-p} [A]_{pp} = C_n (n-p)!x/(x+p)\cdots(x+n) = [A]_{np}$.
Therefore \([A]\) is a Hausdorff matrix. The function \(\alpha(t) = t^x\) on \((0, 1]\) with \(\alpha(0) = 0\) has the property that \(\int_0^1 t^n d\alpha(t) = x/(x+n)\), and hence \([A]\) is a regular matrix.

Suppose (2) is true. If \([A]\) is a Riesz matrix that is also a Hausdorff matrix then \([A]_{(n+1)n} = (n+1)([A]_{nn} - [A]_{(n+1)(n+1)})\) for each \(n\). Moreover \([A]_{(n+1)n} = a_n/S_{(n+1)}\) and the following recursion formula is obtained: \(a_n/S_{n+1} = (n+1)((a_n/S_n) - (a_{(n+1)}/S_{(n+1)}))\). This may be solved for \(a_{n+1}\) and we then obtain the formula \\
\[a_{n+1} = (n/n+1) a_n(S_n/S_{n-1})\], which shows that the sequence \(A\) is completely determined by the term \(a_1\).

If \(a_1 = 0\) then \(A = (1, 0, 0, 0, \ldots)\) and \([A]\) is not regular. It is, however, a semiregular Hausdorff matrix. Hence \(a_1 > 0\). An induction argument shows that \\
\[A = \frac{(1, a_1, a_1(1 + a_1), \ldots, a_1(1 + a_1)(2 + a_1) \cdots (n - 1 + a_1), \ldots)}{2^n!}\]

This concludes the proof of the theorem.

Hausdorff has shown the summability method defined by a Riesz matrix of the type designated in the theorem is equivalent to \((C, 1)\) summability \([1]\).

Reference


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