PRODUCTS OF $k'$-SPACES
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We call a topological space $X$ a $k'$-space if $A \subseteq X, x \in \overline{A}$ implies the existence of a compact subset $K$ of $X$ such that $x \in \text{Cl}(A \cap K)$ (where Cl stands for closure) [1]. A characterization of $k'$-spaces is given in [8]. E. A. Michael has shown that a Hausdorff space is locally compact if its product with every $k'$-space is a $k$-space [6]. We obtain an analogous result for $k'$-spaces which implies the existence of $k'$-spaces which are not $k'$-spaces. It is stated in [1] that such spaces exist as opposed to remarks in [4] and [7]. We show that a $T_1$ space $X$ is discrete if $X \times Y$ is a $k'$-space for every $k'$-space $Y$. Thus a nontrivial product theorem for $k'$-spaces must involve additional conditions on both factors, in contrast to Cohen's Theorem [3] ($X \times Y$ is a $k$-space for $X$ locally compact and $Y$ a $k'$-space—see for instance [2]). We do show that $X \times Y$ is a $k'$-space if both $X$ and $Y$ are $T_1$, $k'$-spaces and $X \times Y$ has a nested neighborhood base at each of its points. Also the product of two $T_1$ spaces with a nested neighborhood base at each point is a $k'$-space if one of the spaces is a $k'$-space and the other is a $k$-space.

**Theorem 1.** If $X$ is a nondiscrete $T_1$ space, then there is a $k'$-space $Y$ such that $X \times Y$ is not a $k'$-space.

**Proof.** Let $\{x_\alpha : \alpha \in D\}$ be a net converging to $x$ such that $x_\alpha \neq x$, $\alpha \in D$. Let $Y_1 = \{(\alpha, n) : \alpha \in D$ and $n = 1, 2, 3, \cdots \}$ and let $Y = Y_1 \cup \{z\}$. The topology on $Y$ is as follows: $Y_1$ is discrete and the open sets containing $z$ contain all but a finite number of elements of each set $A = \{(\alpha, n) : n = 1, 2, 3, \cdots \}$ for $\alpha \in D$. It is easy to see that $Y$ is a $k'$-space since each compact subset intersects only finitely many of the sets $A$. On the other hand $(x, z)$ is a limit point of the set $C = \{(x_\alpha, (\alpha, n)) : \alpha \in D, n = 1, 2, 3, \cdots \}$ but clearly not a limit point of $C \cap K$ for any compact set $K$. Thus $X \times Y$ is not a $k'$-space.

**Remark.** If $X$ is a nondiscrete locally compact $T_1$ space, then there is a $k'$-space $Y$ (as in the proof of the theorem) such that $X \times Y$ is a $k$-space which is not a $k'$-space. The space $Y$ is paracompact. As a matter of fact every open cover has a discrete open refinement. Also $Y$ can be slightly modified to make it a CW-complex.

Before proving Theorem 2 we need a lemma on product spaces with nested neighborhood bases. From this point on we assume that $X$ and $Y$ are $T_1$ spaces.

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Lemma. If \( X \times Y \) has a nested neighborhood base at \( (x, y) \in \overline{A} - A \) and there are neighborhoods \( U \) of \( x \) and \( V \) of \( y \) such that \( \{x\} \times V \cap A = \emptyset \) and \( U \times \{y\} \cap A = \emptyset \), then there is a net \( \{(x_\alpha, y_\alpha) : \alpha \in D\} \) in \( A \) which converges to \( (x, y) \) and, for each \( \alpha_0 \in D \), there are neighborhoods \( R \) of \( x \) and \( S \) of \( y \) such that \( x_\alpha \notin R, y_\alpha \notin S \) for \( \alpha < \alpha_0 \).

Proof. We can choose a nested neighborhood base at \( (x, y) \) of the form \( \{U_\alpha \times V_\alpha : \alpha \in D\} \) where \( D \) is directed as follows: \( \alpha < \beta \) iff \( U_\alpha \times V_\alpha \supset U_\beta \times V_\beta \). We well order the set \( \emptyset = \{U_\alpha \times V_\alpha : \alpha \in D\} \) by \( \prec \) and for each \( \alpha \in D \) choose the first element \( U_\alpha \times V_\alpha \) of \( \emptyset \) such that \( (U_\alpha - U_\beta \times V_\alpha - V_\beta) \cap A \neq \emptyset \). If for some fixed \( \alpha \) no such choice is possible, then \( A \cap (\{U_\alpha - x \times V_\alpha - y \} \cap (x \times V_\alpha - \{x\} \times V_\alpha - \{y\})) \). This contradicts the fact that \( (x, y) \in \overline{A} - A \). Now, for each \( \alpha \in D \), we take \( (x_\alpha, y_\alpha) \in (U_\alpha - U_\beta \times V_\alpha - V_\beta) \cap A \) where \( U_\beta \times V_\beta \) is chosen as indicated above. Let \( \alpha_0 \in D \) and let \( R = U_{\beta_0}, S = V_{\beta_0} \) again choosing \( \beta_0 \) such that \( U_{\beta_0} \times V_{\beta_0} \) is the first element of \( \emptyset \) satisfying \( (U_{\alpha_0} - U_{\beta_0} \times V_{\alpha_0} - V_{\beta_0}) \cap A \neq \emptyset \). We complete the proof by showing that \( x_\alpha \notin U_{\beta_0} \) and \( y_\alpha \notin V_{\beta_0} \) for \( \alpha < \alpha_0 \). Suppose there is \( \alpha < \alpha_0 \) such that \( x_\alpha \in U_{\beta_0} \) or \( y_\alpha \in V_{\beta_0} \). Since \( U_{\beta_0} \times V_{\beta_0} \subset U_{\alpha_0} \times V_{\alpha_0} \subset U_\alpha \times V_\alpha, (U_\alpha - U_{\beta_0} \times V_\alpha - V_{\beta_0}) \cap A \neq \emptyset \). It follows that \( U_\beta \times V_\beta \subset U_{\alpha_0} \times V_{\alpha_0} \). Since \( x_\alpha \notin U_{\beta_0} \) and \( y_\alpha \notin V_{\beta_0} \), we have \( U_\beta \times V_\beta \subset U_{\alpha_0} \times V_{\alpha_0} \). This implies that \( U_{\beta_0} \times V_{\beta_0} \subset U_\beta \times V_\beta \) which is a contradiction. This completes the proof.

By a routine use of the definition of subnet \([5]\) we can establish the fact that any net in \( \{x_\alpha\} \) (resp. \( \{y_\alpha\} \)) which converges to \( x \) (resp. \( y \)) is a subnet of \( \{x_\alpha\} \) (resp. \( \{y_\alpha\} \)). We use this fact in the proof of Theorem 2. We also use the following characterization of \( k \)-spaces established in \([8]\): A topological space \( X \) is a \( k \)-space iff, for each subset \( A \) and \( x \in \overline{A} \), there is a closed \( k \)-subspace \( C \) such that \( x \in \text{Cl}(A \cap C) \).

Theorem 2. If \( X \) is a \( k \)-space and \( Y \) is a \( k \)-space \((k \)-space) and \( X \times Y \) has a nested neighborhood base at each point, then \( X \times Y \) is a \( k \)-space \((k \)-space).

Proof. Let \( A \) be a subset of \( X \times Y \) and let \( x \in \overline{A} - A \). If the neighborhoods \( U \) and \( V \) of the lemma do not exist, then our conclusion follows routinely. Thus there is a net \( \{(x_\alpha, y_\alpha)\} \) in \( A \) converging to \( (x, y) \) and satisfying the conclusion of the lemma. Since \( X \) is a
there is a compact subset $K$ of $X$ such that $x \in \text{Cl}(\{x_\alpha\} \cap K)$.
Thus there is a net $\{x_\gamma\}$ in $\{x_\alpha\} \cap K$ which converges to $x$. By the
note which follows the proof of the lemma $\{x_\gamma\}$ is a subnet of $\{x_\alpha\}$
and $\{y_\gamma\}$ converges to $y$, being a subnet of $\{y_\alpha\}$. Since $Y$ is a $k'$-space,
there is a closed $k$-subspace $C$ of $Y$ (in case $Y$ is a $k'$-space $C$ can be
chosen compact) such that $y \in \text{Cl}(\{y_\gamma\} \cap C)$. Finally we obtain a
subnet of $\{(x_\alpha, y_\alpha)\}$ in $K \times C \cap A$. Thus $(x, y) \in \text{Cl}(K \times C \cap A)$. If $C$
is a $k$-space, then $K \times C$ is a $k$-space [2] and if $C$ is compact $K \times C$
is also. Thus, in case $Y$ is a $k'$-space, $X \times Y$ is a $k$-space and a $k'$-space
when $Y$ is.

**Bibliography**

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