FUTURE STABILITY IS NOT GENERIC

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In [2] S. Smale gave an example to show that structurally stable systems are not dense in the space of all systems. His argument plays two "invariants" against each other: The stable and unstable manifolds. The purpose of this note is to give a new argument for this result; the novelty here is that only one of these invariants is used. Thus "future stable" systems are not dense. Future stability was introduced in a recent lecture of S. Smale in which he expressed some, if not much, hope that it would be a generic property.

We will consider only diffeomorphisms of a manifold. Each diffeomorphism corresponds (via "suspension", e.g. [**1**, p. 797]) to a system on a manifold of one higher dimension. In this setting we give the following:

DEFINITIONS [AFTER SMALE]. For a diffeomorphism $f: M \to M$ we say two points x, $y \in M$ are stably equivalent $x \sim_s y$ provided $\lim_{n\to\infty} \rho(f^n(x), f^n(y)) = 0$. This is an equivalence relation and partitions M into stable sets; the stable set containing x is called $W^s(x, f)$, or $W^s(x)$. $W^u(x, f)$ is by definition $W^s(x, f^{-1})$. f is *future stable* if for all nearby pertubations f' of f there is a (topological) homeomorphism $h: M \to M$ which sends stable sets of f into stable sets of f'; that is, $h(W^s(x, f)) = W^s(h(x), f')$, for all $x \in M$.

We also need a more technical definition. Suppose some of the stable sets of f are smooth manifolds of codimension one, which smoothly foliate an open set. Then locally they are the level sets of a real valued function, say $\phi: U \to \mathbf{R}$. One says that an unstable set $W^u(p)$ *hooks at* $W^s(q)$ provided that there is $x_0 \in W^u(p) \cap W^s(q) \cap U$ where $\phi(x_0)$ is an isolated local maximum of the function $W^u(p) \to \mathbf{R}$ given by $x \to \phi(x)$. Let T^2 be a two-dimensional torus.

THEOREM. *There is an open set* $\mathfrak{U} \subset \mathrm{Diff}^r(T^2)$, $r \geqq 2$, *and dense subsets* $\mathfrak{A}$, $\mathfrak{B}$ *of* $\mathfrak{U}$ *such that each* $f \in \mathfrak{U}$ *has an attractor* Σ *(a* 1*-dimensional generalized solenoid) and a hyperbolic fixed point* p, *where*

1. $W^u(x)$, $W^s(x)$ *are smooth copies of* $\mathbf{R}$, *for* $x \in \Sigma$.
2. $\Sigma = \bigcup_{x\in\Sigma} W^u(x)$.
3. *A neighborhood of* Σ *is foliated by* $\{W^s(x) : x \in \Sigma\}$.
4. $W^u(p)$ *hooks at, and only at,* $W^s(q)$, *where*

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(a) *q is a periodic point, if $f\in\mathfrak{A}$;*
(b) *q is aperiodic and has a dense orbit, if $f\in\mathfrak{B}$.*

This result is due to S. Smale, who proved it (with appropriate modification) for T^3; for T^2, see [3]. It is also true for the C^1-topology if one allows finitely many points to play the role of q.

Now for $f\in\mathfrak{U}$, let $B(f)$ be the compact set $p\cup W^u(p)\cup\Sigma(f)$. $B(f)$ is the union of a certain solenoid, the point p and a line, $W^u(p)$.

Claim. If $f\in\mathfrak{U}$, then f^{-1} is not future stable. For otherwise there would be nonempty open sets $\mathfrak{V}\subset\mathfrak{U}$ such that each two $B(f)$, $f\in\mathfrak{V}$ would be homeomorphic. But

LEMMA. *For $f\in\mathfrak{A}$, $g\in\mathfrak{B}$, $B(f)$ and $B(g)$ are not homeomorphic.*

PROOF. Let S be the closure of the graph of $\sin(1/x)$, $x\in\boldsymbol{R}$. Then the line interval joining (0, 1) to (0, -1) is in S. Now finitely many (as many as the period of q) points of $\Sigma(f)$ have small neighborhoods which contain a topological copy of a small neighborhood in S of (0, 1). All other points of $\Sigma(f)$ have small neighborhoods of the form (some set)$\times[0, 1]$, i.e. no arc doubles back, near them.

However, as the points in $B(g)$ which correspond to points at which $W^u(p)$ *hooks at* $W^s(q)$, converge to all of $\Sigma(g)$, $\Sigma(g)$ has no point of this second type.

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