

FUTURE STABILITY IS NOT GENERIC

MICHAEL SHUB AND R. F. WILLIAMS¹

In [2] S. Smale gave an example to show that structurally stable systems are not dense in the space of all systems. His argument plays two “invariants” against each other: The stable and unstable manifolds. The purpose of this note is to give a new argument for this result; the novelty here is that only one of these invariants is used. Thus “future stable” systems are not dense. Future stability was introduced in a recent lecture of S. Smale in which he expressed some, if not much, hope that it would be a generic property.

We will consider only diffeomorphisms of a manifold. Each diffeomorphism corresponds (via “suspension”, e.g. [1, p. 797]) to a system on a manifold of one higher dimension. In this setting we give the following:

DEFINITIONS [AFTER SMALE]. For a diffeomorphism $f: M \rightarrow M$ we say two points $x, y \in M$ are stably equivalent $x \sim_s y$ provided $\lim_{n \rightarrow \infty} \rho(f^n(x), f^n(y)) = 0$. This is an equivalence relation and partitions M into stable sets; the stable set containing x is called $W^s(x, f)$, or $W^s(x)$. $W^u(x, f)$ is by definition $W^s(x, f^{-1})$. f is *future stable* if for all nearby perturbations f' of f there is a (topological) homeomorphism $h: M \rightarrow M$ which sends stable sets of f into stable sets of f' ; that is, $h(W^s(x, f)) = W^s(h(x), f')$, for all $x \in M$.

We also need a more technical definition. Suppose some of the stable sets of f are smooth manifolds of codimension one, which smoothly foliate an open set. Then locally they are the level sets of a real valued function, say $\phi: U \rightarrow \mathbf{R}$. One says that an unstable set $W^u(p)$ *hooks at* $W^s(q)$ provided that there is $x_0 \in W^u(p) \cap W^s(q) \cap U$ where $\phi(x_0)$ is an isolated local maximum of the function $W^u(p) \rightarrow \mathbf{R}$ given by $x \mapsto \phi(x)$. Let T^2 be a two-dimensional torus.

THEOREM. *There is an open set $\mathcal{U} \subset \text{Diff}^r(T^2)$, $r \geq 2$, and dense subsets \mathcal{G}, \mathcal{B} of \mathcal{U} such that each $f \in \mathcal{U}$ has an attractor Σ (a 1-dimensional generalized solenoid) and a hyperbolic fixed point p , where*

1. $W^u(x), W^s(x)$ are smooth copies of \mathbf{R} , for $x \in \Sigma$.
2. $\Sigma = \bigcup_{x \in \Sigma} W^u(x)$.
3. A neighborhood of Σ is foliated by $\{W^s(x) : x \in \Sigma\}$.
4. $W^u(p)$ hooks at, and only at, $W^s(q)$, where

Received by the editors December 19, 1968.

¹ The first author was partially supported during the period this research was done by Army contract DA-ARO-D-31-124-G866, the second by NSF Grant 5591.

- (a) q is a periodic point, if $f \in \mathcal{A}$;
- (b) q is aperiodic and has a dense orbit, if $f \in \mathcal{B}$.

This result is due to S. Smale, who proved it (with appropriate modification) for T^3 ; for T^2 , see [3]. It is also true for the C^1 -topology if one allows finitely many points to play the role of g .

Now for $f \in \mathcal{U}$, let $B(f)$ be the compact set $p \cup W^u(p) \cup \Sigma(f)$. $B(f)$ is the union of a certain solenoid, the point p and a line, $W^u(p)$.

Claim. If $f \in \mathcal{U}$, then f^{-1} is not future stable. For otherwise there would be nonempty open sets $\mathcal{V} \subset \mathcal{U}$ such that each two $B(f), f \in \mathcal{V}$ would be homeomorphic. But

LEMMA. *For $f \in \mathcal{A}, g \in \mathcal{B}$, $B(f)$ and $B(g)$ are not homeomorphic.*

PROOF. Let S be the closure of the graph of $\sin(1/x)$, $x \in \mathbb{R}$. Then the line interval joining $(0, 1)$ to $(0, -1)$ is in S . Now finitely many (as many as the period of g) points of $\Sigma(f)$ have small neighborhoods which contain a topological copy of a small neighborhood in S of $(0, 1)$. All other points of $\Sigma(f)$ have small neighborhoods of the form (some set) $\times [0, 1]$, i.e. no arc doubles back, near them.

However, as the points in $B(g)$ which correspond to points at which $W^u(p)$ hooks at $W^s(q)$, converge to all of $\Sigma(g)$, $\Sigma(g)$ has no point of this second type.

BIBLIOGRAPHY

1. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747.
2. S. Smale, *Structurally stable systems are not dense*, Amer. J. Math. **88** (1966), 491.
3. R. Williams, *The D.A. maps of Smale and structural stability*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I. (to appear).

BRANDEIS UNIVERSITY, NORTHWESTERN UNIVERSITY AND
UNIVERSITY OF GENEVA