

ON PROJECTIVE MODULES OF FINITE RANK

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One of the aims of this paper is to answer the following question: Let R be a commutative ring for which projective ideals are finitely generated; is the same valid in $R[x]$, the polynomial ring in one variable over R ? A Hilbert basis type of argument does not seem to lead directly to a solution. Instead we were taken to consider a special case of the following problem: Let (X, O_X) be a prescheme and M a quasi-coherent O_X -module with finitely generated stalks; when is M of finite type? Examples abound where this is not so and here it is shown that a ring for which a projective module with finitely generated localizations is always finitely generated, is precisely one of the kind mentioned above (Theorem 2.1). Such a ring R could also be characterized as "any finitely generated flat module is projective."

1. Preliminaries. Throughout R will denote a commutative ring. Let M be a projective R -module; for each prime \mathfrak{p} , $M_{\mathfrak{p}}$, the localization of M with respect to $R - \mathfrak{p}$, is by [4] a free $R_{\mathfrak{p}}$ -module, with a basis of cardinality $\rho_M(\mathfrak{p})$. ρ_M is the so-called rank function of M and here it will be assumed that ρ_M takes only finite values.

The trace of M is defined to be the image of the map $M \otimes_R \text{Hom}_R(M, R) \rightarrow R$, $m \otimes f \rightarrow f(m)$; it is denoted by $\tau_R(M)$. If $M \oplus N = F$ (free), it is clear that $\tau_R(M)$ is the ideal of R generated by the coordinates of all elements in M , for any basis chosen in F . It follows that for any homomorphism $R \rightarrow S$, $\tau_S(M \otimes_R S) = \tau_R(M)S$. In particular, if \mathfrak{p} is a prime ideal, $\tau_R(M)_{\mathfrak{p}} = \tau_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = (0)$ or (1) depending on whether $M_{\mathfrak{p}} = (0)$ or $M_{\mathfrak{p}} \supseteq (0)$. Thus $(\tau_R(M))^2 = \tau_R(M)$ and $\tau_R(M)$ is generated by an idempotent if it is finitely generated; also, $\tau_R(M)M = M$. For later reference, we isolate part of this discussion into

(1.1) LEMMA. *The support of a projective module M is open in $\text{Spec } R$.*

PROOF. In fact, it was shown that $\text{Supp } M = \{\text{all primes not containing } \tau_R(M)\}$. (See also [5].)

In its simplest case the trace and the finite generation of a projective module are related by

(1.2) LEMMA. *Let M be a projective module such that for each prime*

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\mathfrak{p} , $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ or (0) . Then M is finitely generated iff $\tau_R(M)$ is finitely generated.

PROOF. If $\tau_R(M)$ is finitely generated, as remarked earlier, it can be generated by an idempotent $\tau_R(M) = Re$ and M can be viewed as a projective module over Re with unit trace. Changing notation can write

$$(*) \quad \sum_{i=1}^n f_i(m_i) = 1$$

with $f_i \in \text{Hom}_R(M, R)$. Now, for each maximal ideal \mathfrak{p} , $M/\mathfrak{p}M \cong R/\mathfrak{p}$ and an element of M generates $M_{\mathfrak{p}}$ provided it is in $M - \mathfrak{p}M$. As $(*)$ makes it impossible for all x_i 's to be in $\mathfrak{p}M$ it is clear that x_1, \dots, x_n generate M for they do so locally. The converse is trivial.

This can be extended to higher ranks through the device of exterior products. If M is a projective module its exterior powers are also projective as $M \oplus N = F(\text{free})$ implies

$$\bigwedge^r (M \oplus N) = \bigwedge^r M \oplus \dots \oplus \bigwedge^r N = \bigwedge^r F$$

and the exterior powers of a free module are clearly free [3].

(1.3) PROPOSITION. Let M be a projective module of constant finite rank (i.e. $\rho_M(\mathfrak{p}) = r$ for any prime \mathfrak{p} , r a fixed integer); then M is finitely generated.

PROOF. If r is the integer above, then $\bigwedge^r M$ is a projective module of rank 1 and by (1.2) finitely generated. Pick m_1, \dots, m_n elements in M such that their r -exterior products generate $\bigwedge^r M$. To check that those elements generate M it is enough to do it locally when it becomes clear.

Finally, if the rank function is no longer constant, one has

(1.4) PROPOSITION. Let M be a projective module with $M_{\mathfrak{p}}$ finitely generated over $R_{\mathfrak{p}}$ and bounded rank. Then M is finitely generated iff ρ_M is locally constant.

PROOF. The limitation on the ranks of the localizations of M can be expressed by saying that there is a last integer r for which $\bigwedge^r M \neq (0)$. Let I_r be the trace of $\bigwedge^r M$. As ρ_M is locally constant, $\text{supp } \bigwedge^r M$ is then both open and closed. There is then an ideal J_r with $I_r + J_r = R$, $I_r \cap J_r = \text{nilideal}$. Through standard techniques, an idempotent e can be found in I_r such that $(e) + J_r = R$. As I_r is locally (0) or (1) it is easy to verify that $I_r = (e)$. Decompose R into

$Re \oplus R(1-e)$ and, similarly, $M = eM \oplus (1-e)M$. As an Re -module, eM is projective of constant rank r . As for $(1-e)M$, it has rank $< r$ as a projective $R(1-e)$ -module and inherits the rank function of M on $\text{Spec}(R(1-e))$. (1.3) and induction finish the proof.

2. Main results. The question raised at the outset is now answered. It will follow from

(2.1) THEOREM. *Let R be a commutative ring. The following are equivalent for R -modules:*

- (1) *Finitely generated flat modules are projective.*
- (2) *A locally finitely generated projective module is finitely generated.*
- (3) *Projective ideals are finitely generated.*

PROOF. (1) \Rightarrow (2): Let M be a projective module. If I_r denotes the trace of $\wedge^r M$, then R/I_r is a flat module and thus projective, and I_r is generated by an idempotent. If ρ_M is bounded we are done by (1.4); if not, we get a decreasing sequence $I_1 \supseteq I_2 \supseteq \cdots$ of ideals generated by idempotents, say $I_i = (e_i)$. Consider the sequence

$$(f_1) \subseteq (f_2) \subseteq \cdots, \quad f_i = 1 - e_i.$$

Let $I = \bigcup (f_i)$. $I \neq R$ as no $e_i = 0$. But I is locally (0) or (1) and, as before, going over to R/I , we conclude that I is finitely generated, i.e., $I = (f_n)$ for some n . But for any prime $\mathfrak{p} \supseteq (f_n)$ we get that $\rho_M(\mathfrak{p})$ is nonfinite.

(2) \Rightarrow (3): Obvious.

(3) \Rightarrow (1): For this we use an argument inspired by [1]. It is enough to show that cyclic flat modules are projective [6], [7]. Let R/I be one such, $I \neq 0$. Then I is locally (0) or (1). Let a_1 be a nonzero element of I ; then $(a_1) = a_1 I$, checked by localization. Write $a_1 = a_1 a_2$, $a_2 \in I$. Proceeding in this fashion we get a sequence of principal ideals $(a_1) \subseteq (a_2) \subseteq \cdots$ with $a_i = a_i a_{i+1}$. If it becomes stationary at (a_n) , say, $a_{n+1} = \lambda a_n$ and $a_{n+1}^2 = a_{n+1} \lambda a_n = \lambda a_n = a_{n+1}$. If $(a_n) \neq I$, we split $I = (a_n) \oplus I'$, with I' also being locally (0) or (1). Doing the same for I' we get a similar sequence $(a'_1) \subseteq (a'_2) \subseteq \cdots$, and thus a longer sequence

$$(a_1) \subseteq (a_2) \subseteq \cdots \subseteq (a_n) \subseteq (a_n + a'_1) \subseteq (a_n + a'_2) \subseteq \cdots$$

In this way either I turns out to be generated by an idempotent (as wanted) or we get a strictly increasing sequence $(x_1) \subseteq (x_2) \subseteq \cdots$ with $x_i = x_i x_{i+1}$. Assume this is the case; write $J = \bigcup (x_i)$. By exhibiting a "dual basis" for J [2, p. 132] we prove J to be projective. Let $f_i \in \text{Hom}_R(J, R)$ be defined: $f_1(x) = x$; $f_i(x) = (1 - x_{i-1})x$, $i > 1$. We

claim that $f_i(x) = 0$ for all large i 's. If, say, $x = \lambda x_{n-2}$, $f_n(x) = (1 - a_{n-1})\lambda a_{n-2} = 0$. Also, $\sum f_i(x)x_i = x$. This contradicts (3) and I is finitely generated.

We can now state

(2.2) COROLLARY. *Let R be a ring for which projective ideals are finitely generated. Then the polynomial ring, $R[x]$, enjoys the same property.*

PROOF. It is enough to show that the support of a projective module is closed. For a projective $R[x]$ -module M one checks easily that $\text{Supp } M$ is defined by the ideal $\tau_R(M/xM) \cdot R[x]$. As $\tau_R(M/xM)$ is generated by an idempotent, we are through.

3. **Remarks.** It is not always the case that the rank of a locally finitely generated projective is bounded. Consider R to be a regular (von Neumann) ring with countably many prime ideals (e.g. $R = \text{subring of } \prod^{\infty} k$, k a field, generated by ke , $e = \text{identity}$ and the basic idempotents). Number the prime ideals: $\mathfrak{m}_0, \mathfrak{m}_1, \dots$. Each is projective. Let

$$M = \mathfrak{m}_0 \oplus \mathfrak{m}_0 \cdot \mathfrak{m}_1 \oplus \dots \oplus \mathfrak{m}_0 \cdot \mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_n \oplus \dots ;$$

it is projective and $M_{\mathfrak{m}_n} \cong (R_{\mathfrak{m}_n})^n$.

What are the rings of (2.1)? Obviously domains, semilocal rings, polynomial or power series over them. If R is semihereditary, it must be a direct sum of Prufer domains [7]. It seems plausible that if R satisfies the conditions of (2.1), the same holds in $R[x_\alpha \text{'s}] = \text{polynomial ring in any number of variables}$.

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