ON PROJECTIVE MODULES OF FINITE RANK

WOLMER V. VASCONCELOS[1]

One of the aims of this paper is to answer the following question: Let R be a commutative ring for which projective ideals are finitely generated; is the same valid in $R[x]$, the polynomial ring in one variable over R? A Hilbert basis type of argument does not seem to lead directly to a solution. Instead we were taken to consider a special case of the following problem: Let (X, O_X) be a prescheme and M a quasi-coherent O_X-module with finitely generated stalks; when is M of finite type? Examples abound where this is not so and here it is shown that a ring for which a projective module with finitely generated localizations is always finitely generated, is precisely one of the kind mentioned above (Theorem 2.1). Such a ring R could also be characterized as "any finitely generated flat module is projective."

1. **Preliminaries.** Throughout R will denote a commutative ring. Let M be a projective R-module; for each prime $\mathfrak{p}$, $M_{\mathfrak{p}}$, the localization of M with respect to $R-\mathfrak{p}$, is by [4] a free $R_{\mathfrak{p}}$-module, with a basis of cardinality $\rho_M(\mathfrak{p})$. ρ_M is the so-called rank function of M and here it will be assumed that ρ_M takes only finite values.

The trace of M is defined to be the image of the map $M \otimes_R \operatorname{Hom}_R(M, R) \to R$, $m \otimes f \to f(m)$; it is denoted by $\tau_R(M)$. If $M \oplus N = F$(free), it is clear that $\tau_R(M)$ is the ideal of R generated by the coordinates of all elements in M, for any basis chosen in F. It follows that for any homorphism $R \to S$, $\tau_S(M \otimes_R S) = \tau_R(M)S$. In particular, if $\mathfrak{p}$ is a prime ideal, $\tau_R(M)_{\mathfrak{p}} = \tau_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = (0)$ or (1) depending on whether $M_{\mathfrak{p}} = (0)$ or $M_{\mathfrak{p}} \supsetneq (0)$. Thus $(\tau_R(M))^2 = \tau_R(M)$ and $\tau_R(M)$ is generated by an idempotent if it is finitely generated; also, $\tau_R(M)M = M$. For later reference, we isolate part of this discussion into

(1.1) LEMMA. *The support of a projective module M is open in Spec R.*

PROOF. In fact, it was shown that Supp $M = \{$all primes not containing $\tau_R(M)\}$. (See also [5].)

In its simplest case the trace and the finite generation of a projective module are related by

(1.2) LEMMA. *Let M be a projective module such that for each prime*

Received by the editors November 15, 1968.

[1] The author gratefully acknowledges partial support from the National Science Foundation under grant GP-8619.

$\boldsymbol{p}$, $M_{\boldsymbol{p}} \cong R_{\boldsymbol{p}}$ *or* (0). *Then* M *is finitely generated iff* $\tau_R(M)$ *is finitely generated.*

PROOF. If $\tau_R(M)$ is finitely generated, as remarked earlier, it can be generated by an idempotent $\tau_R(M) = Re$ and M can be viewed as a projective module over Re with unit trace. Changing notation can write

$$(*) \qquad \sum_{i=1}^{n} f_i(m_i) = 1$$

with $f_i \in \mathrm{Hom}_R(M, R)$. Now, for each maximal ideal $\boldsymbol{p}$, $M/\boldsymbol{p}M \cong R/\boldsymbol{p}$ and an element of M generates $M_{\boldsymbol{p}}$ provided it is in $M - \boldsymbol{p}M$. As (*) makes it impossible for all x_i's to be in $\boldsymbol{p}M$ it is clear that $x_1, \cdots, x_n$ generate M for they do so locally. The converse is trivial.

This can be extended to higher ranks through the device of exterior products. If M is a projective module its exterior powers are also projective as $M \oplus N = F$(free) implies

$$\bigwedge^r (M \oplus N) = \bigwedge^r M \oplus \cdots \oplus \bigwedge^r N = \bigwedge^r F$$

and the exterior powers of a free module are clearly free [3].

(1.3) PROPOSITION. *Let* M *be a projective module of constant finite rank (i.e.* $\rho_M(\boldsymbol{p}) = r$ *for any prime* $\boldsymbol{p}$, r *a fixed integer); then* M *is finitely generated.*

PROOF. If r is the integer above, then $\wedge^r M$ is a projective module of rank 1 and by (1.2) finitely generated. Pick $m_1, \cdots, m_n$ elements in M such that their r-exterior products generate $\wedge^r M$. To check that those elements generate M it is enough to do it locally when it becomes clear.

Finally, if the rank function is no longer constant, one has

(1.4) PROPOSITION. *Let* M *be a projective module with* $M_{\boldsymbol{p}}$ *finitely generated over* $R_{\boldsymbol{p}}$ *and bounded rank. Then* M *is finitely generated iff* ρ_M *is locally constant.*

PROOF. The limitation on the ranks of the localizations of M can be expressed by saying that there is a last integer r for which $\wedge^r M \neq (0)$. Let I_r be the trace of $\wedge^r M$. As ρ_M is locally constant, supp $\wedge^r M$ is then both open and closed. There is then an ideal J_r with $I_r + J_r = R$, $I_r \cap J_r =$ nilideal. Through standard techniques, an idempotent e can be found in I_r such that $(e) + J_r = R$. As I_r is locally (0) or (1) it is easy to verify that $I_r = (e)$. Decompose R into

$Re\oplus R(1-e)$ and, similarly, $M=eM\oplus(1-e)M$. As an Re-module, eM is projective of constant rank r. As for $(1-e)M$, it has rank $<r$ as a projective $R(1-e)$-module and inherits the rank function of M on $\operatorname{Spec}(R(1-e))$. (1.3) and induction finish the proof.

2. **Main results.** The question raised at the outset is now answered. It will follow from

(2.1) THEOREM. *Let R be a commutative ring. The following are equivalent for R-modules:*

(1) *Finitely generated flat modules are projective.*

(2) *A locally finitely generated projective module is finitely generated.*

(3) *Projective ideals are finitely generated.*

PROOF. (1)⇒(2): Let M be a projective module. If I_r denotes the trace of $\wedge^r M$, then R/I_r is a flat module and thus projective, and I_r is generated by an idempotent. If ρ_M is bounded we are done by (1.4); if not, we get a decreasing sequence $I_1\supseteq I_2\supseteq\cdots$ of ideals generated by idempotents, say $I_i=(e_i)$. Consider the sequence

$$(f_1)\subseteq(f_2)\subseteq\cdots,\qquad f_i=1-e_i.$$

Let $I=\bigcup(f_i)$. $I\neq R$ as no $e_i=0$. But I is locally (0) or (1) and, as before, going over to R/I, we conclude that I is finitely generated, i.e., $I=(f_n)$ for some n. But for any prime $\mathfrak{p}\supseteq(f_n)$ we get that $\rho_M(\mathfrak{p})$ is nonfinite.

(2)⇒(3): Obvious.

(3)⇒(1): For this we use an argument inspired by [1]. It is enough to show that cyclic flat modules are projective [6], [7]. Let R/I be one such, $I\neq 0$. Then I is locally (0) or (1). Let a_1 be a nonzero element of I; then $(a_1)=a_1I$, checked by localization. Write $a_1=a_1a_2$, $a_2\in I$. Proceeding in this fashion we get a sequence of principal ideals $(a_1)\subseteq(a_2)\subseteq\cdots$ with $a_i=a_ia_{i+1}$. If it becomes stationary at (a_n), say, $a_{n+1}=\lambda a_n$ and $a_{n+1}^2=a_{n+1}\lambda a_n=\lambda a_n=a_{n+1}$. If $(a_n)\neq I$, we split $I=(a_n)\oplus I'$, with I' also being locally (0) or (1). Doing the same for I' we get a similar sequence $(a_1')\subseteq(a_2')\subseteq\cdots$, and thus a longer sequence

$$(a_1)\subseteq(a_2)\subseteq\cdots\subseteq(a_n)\subseteq(a_n+a_1')\subseteq(a_n+a_2')\subseteq\cdots.$$

In this way either I turns out to be generated by an idempotent (as wanted) or we get a strictly increasing sequence $(x_1)\subseteq(x_2)\subseteq\cdots$ with $x_i=x_ix_{i+1}$. Assume this is the case; write $J=\bigcup(x_i)$. By exhibiting a "dual basis" for J [2, p. 132] we prove J to be projective. Let $f_i\in\operatorname{Hom}_R(J,R)$ be defined: $f_1(x)=x$; $f_i(x)=(1-x_{i-1})x$, $i>1$. We claim that $f_i(x)=0$ for all large i's. If, say, $x=\lambda x_{n-2}$, $f_n(x) = (1-a_{n-1})\lambda a_{n-2}=0$. Also, $\sum f_i(x)x_i=x$. This contradicts (3) and I is finitely generated.

We can now state

(2.2) COROLLARY. *Let R be a ring for which projective ideals are finitely generated. Then the polynomial ring, $R[x]$, enjoys the same property.*

PROOF. It is enough to show that the support of a projective module is closed. For a projective $R[x]$-module M one checks easily that Supp M is defined by the ideal $\tau_R(M/xM)\cdot R[x]$. As $\tau_R(M/xM)$ is generated by an idempotent, we are through.

3. **Remarks.** It is not always the case that the rank of a locally finitely generated projective is bounded. Consider R to be a regular (von Neumann) ring with countably many prime ideals (e.g. $R=$subring of $\prod^\infty k$, k a field, generated by ke, $e=$identity and the basic idempotents). Number the prime ideals: $\mathfrak{m}_0, \mathfrak{m}_1, \cdots$. Each is projective. Let

$$M = \mathfrak{m}_0 \oplus \mathfrak{m}_0\cdot\mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_0\cdot\mathfrak{m}_1\cdots\mathfrak{m}_n \oplus \cdots;$$

it is projective and $M_{\mathfrak{m}_n}\cong(R_{\mathfrak{m}_n})^n$.

What are the rings of (2.1)? Obviously domains, semilocal rings, polynomial or power series over them. If R is semihereditary, it must be a direct sum of Prufer domains [7]. It seems plausible that if R satisfies the conditions of (2.1), the same holds in $R[x_\alpha$'s$]=$polynomial ring in any number of variables.

BIBLIOGRAPHY

1. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–492.

2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

3. C. Chevalley, *Fundamental concepts of algebra*, Academic Press, New York, 1956.

4. I. Kaplansky, *Projective modules*, Ann. of Math. **68** (1958), 372–377.

5. D. Lazard, *Disconnexités des spectres d'anneaux et des préschémas*, Bull. Soc. Math. France **95** (1967), 95–108.

6. K. Mount, *Some remarks on Fitting's invariants*, Pacific J. Math. **13** (1963), 1453–1357.

7. W. V. Vasconcelos, *On finitely generated flat modules*, Trans. Amer. Math. Soc. **138** (1969), 505–512.

RUTGERS UNIVERSITY