NONSTANDARD ALMOST PERIODIC FUNCTIONS ON A GROUP[1]

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J. von Neumann's theory of almost periodic functions on an arbitrary group [3] begins with a definition of almost periodicity which was first formulated for functions on the group of real numbers by S. Bochner [1]. In the present paper, a nonstandard form of this definition is used to obtain two other characterizations of almost periodicity, each of which generalizes H. Bohr's original definition involving translation numbers. These results form the basis for a description of what might be called the Bohr compactification of a group with respect to an almost periodic function. Finally, a nonstandard proof is given of the result, essentially due to A. Weil [5], that the class of continuous functions on the compactification is isometrically isomorphic to a certain class of almost periodic functions.

We assume familiarity with the properties of an enlargement of a mathematical structure, in the sense of A. Robinson, and with the notation of [4].

Let G be a group with identity e and let $C(G)$ denote the algebra of complex valued functions on G under the supnorm. A set of functions from $C(G)$ is called conditionally compact (c.c.) if every sequence of functions taken from the set contains a fundamental subsequence.

DEFINITION 1. A function f in $C(G)$ is called right almost periodic (r.a.p.) if the set R_f of all functions $f(xa)$ (where a is a parameter running over G) is c.c.; it is called left almost periodic (l.a.p.) if the set L_f of all functions $f(ax)$ is c.c.; it is called almost periodic (a.p.) if it is r.a.p. and l.a.p.

This is von Neumann's definition, from which it is not hard to show that r.a.p. and l.a.p. functions are bounded. We therefore restrict our attention to the metric space $B(G)$ of bounded functions on G.

According to nonstandard metric space theory, conditional compactness can be characterized as follows: a subset N of a metric space M is conditionally compact if and only if for every element n in the enlargement $^*N$ of N, there is a standard $m \in M$ whose distance from n is infinitesimal [4, pp. 94, 102]. It follows that the nonstandard analogue of Definition 1 is given by

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THEOREM 1. *A function f in $B(G)$ is r.a.p. (l.a.p.) if and only if for any element a of the enlargement $^*G$ of G, there is a function $g_a(x)$ $(h_a(x))$ in $B(G)$ such that for all $x\in{}^*G$, $g_a(x)\simeq f(xa)$ $(h_a(x)\simeq f(ax))$.*

The first of the two characterizations of almost periodicity given below is well known and involves the notion of a relatively dense set. The proof is nonstandard, and rests on a simple nonstandard condition for relative density.

DEFINITION 2. A set $E\subset G$ is said to be right relatively dense (r.r.d) if there exist $r_1, \cdots, r_n\in G$ such that $\bigcup_{i=1}^n r_iE=G$; E is left relatively dense (l.r.d.) if there exist $r_1, \cdots, r_n\in G$ such that $\bigcup_{i=1}^n Er_i=G$; E is relatively dense (r.d.) if it is r.r.d. and l.r.d.

THEOREM 2. *A set $E\subset G$ is r.r.d. if and only if for every $a\in{}^*G$, $G\cap{}^*Ea\neq\varnothing$; E is l.r.d. if and only if for every $a\in{}^*G$, $G\cap a^*E\neq\varnothing$.*

PROOF. Suppose first that E is not r.r.d., i.e., that for any $r_1, \cdots, r_n\in G$, there exists $a\in G$ such that $a\notin r_iE$, $i=1, 2, \cdots, n$. Then the relation $R(r, a)\leftrightarrow a\notin rE$ is concurrent in G, so by definition of an enlargement, there exists $a\in{}^*G$ such that for all $r\in G$, $a\notin r^*E$, or equivalently, $r^{-1}\notin{}^*Ea^{-1}$. Thus $G\cap{}^*Ea^{-1}=\varnothing$.

Conversely, let $r_1, \cdots, r_n\in G$, and suppose there exists $a\in{}^*G$ such that $G\cap{}^*Ea=\varnothing$. Then $r_i\notin{}^*Ea$, $i=1, 2, \cdots, n$. By transfer to G, there exists $a\in G$ such that $r_i\notin Ea$, $i=1, 2, \cdots, n$, so that $\bigcup_{i=1}^n r_i^{-1}E\neq G$. The first part of the theorem follows by contraposition. The proof of the second part is similar and will be omitted.

Given a function $f\in B(G)$ and any standard real number $\epsilon>0$, let $E(\epsilon, f)=\{t\in G \mid \text{for all } y, z, \in G, \ |f(ytz)-f(yz)|<\epsilon\}$. The elements of the sets $E(\epsilon, f)$ are called translation elements for f.

THEOREM 3. *A function $f\in B(G)$ is r.a.p. (l.a.p.) if and only if $E(\epsilon, f)$ is r.r.d. (l.r.d.) for every $\epsilon>0$. f is a.p. if and only if $E(\epsilon, f)$ is r.d. for every $\epsilon>0$.*

PROOF. We consider only the r.a.p. case. To prove sufficiency, let $\epsilon>0$ be standard and for $a\in{}^*G$, define $g_a\in B(G)$ by $g_a(u)={}^0(f(ua))$. For any $t\in E(\epsilon/3, f)$, the statement $(\forall z)[|f(etz)-f(ez)|<\epsilon/3]$ holds in $B(G)$ and hence by transfer it holds in $^*B(G)$. Thus for all $x\in G$,

$$|g_a(tx)-g_a(x)| \simeq |f(txa)-f(xa)| < \epsilon/3,$$

from which it follows that

$$(\forall t\in{}^*E(\epsilon/3, f))(\forall x\in{}^*G)[|g_a(tx)-g_a(x)| < \epsilon/2]$$

holds in $^*B(G)$. Now for any $x\in{}^*G$, $^*E(\epsilon/3, f)x\cap G\neq\varnothing$ by Theorem

2, so there exists $t\in {}^*E(\epsilon/3,f)$ such that $tx\in G$. Now we have

$$|f(xa)-g_a(x)| \leqq |f(xa)-f(txa)| + |f(txa)-g_a(tx)| + |g_a(tx)-g_a(x)| < \epsilon$$

which proves that $f(xa)\simeq g_a(x)$ for all $x\in {}^*G$. By Theorem 1, f is r.a.p.

To prove necessity, suppose that there exists a standard $\epsilon>0$, for which $E(\epsilon,f)$ is not r.r.d. Then for some $a\in {}^*G$, $G\cap {}^*E(\epsilon,f)a=\varnothing$, by Theorem 2. Moreover, if $u\in G$ and $t\in Ga^{-1}$, then $u^{-1}t\in Ga^{-1}$, and there exists $y\in {}^*G$ such that $|f(yu^{-1}te)-f(ye)|\geqq\epsilon$. On the other hand, there is by Theorem 1 a standard function $g_t\in B(G)$ such that $g_t(yu^{-1})\simeq f(yu^{-1}t)$ for all $y\in {}^*G$. Hence for each $t\in Ga^{-1}$ and $u\in G$, the statement

$$(\exists y)[|g_t(yu^{-1})-f(y)|\geqq\epsilon/2] \tag{1}$$

holds in ${}^*B(G)$. Since g_t and u are standard, (1) is defined and true in $B(G)$. It follows that for each $t\in Ga^{-1}$, the statement

$$(\forall u)(\exists y)[|g_t(yu^{-1})-f(y)|\geqq\epsilon/2] \tag{2}$$

holds in $B(G)$ and, by transfer, in ${}^*B(G)$. Now (2) can be particularized by assigning to u the value t, so for some $y\in {}^*G$,

$$|g_t(yt^{-1})-f(y)| = |g_t(yt^{-1})-f(yt^{-1}t)| \geqq \epsilon/2.$$

This inequality contradicts the characteristic property of g_t and completes the proof.

We come now to a nonstandard characterization of almost periodicity. For $f\in B(G)$, let $E(f)=\{t\in {}^*G \mid \text{for all } x, y\in {}^*G, f(xty)\simeq f(xy)\}$.

THEOREM 4. *A function $f\in B(G)$ is r.a.p. (l.a.p.) if and only if for every infinite positive integer N there exist $r_1,\cdots,r_N\in {}^*G$ such that $\bigcup_{i=1}^N r_iE(f)={}^*G$ $(\bigcup_{i=1}^N E(f)r_i={}^*G)$.*

PROOF. *Sufficiency.* If $\epsilon>0$ is standard, the sentence "there exist $r_1,\cdots,r_N\in {}^*G$ such that $\bigcup_{i=1}^N r_i{}^*E(\epsilon,f)={}^*G$" holds in ${}^*B(G)$ because ${}^*E(\epsilon,f)\supset E(f)$. The sentence is therefore true in $B(G)$, where it asserts that $E(\epsilon,f)$ is r.r.d. Thus by Theorem 3, f is r.a.p.

Necessity. Let $\epsilon>0$. By Theorem 3, there exist $r_1,\cdots,r_n\in G$ such that $\bigcup_{i=1}^n r_iE(\epsilon,f)=G$. Let $n(\epsilon)$ be the smallest integer n for which this condition is satisfied. Then $n(\epsilon)$ is clearly a decreasing function of ϵ, so that ${}^*n(\epsilon)$ is finite for all positive $\epsilon\not\simeq 0$. It follows that if N is any infinite positive integer, there is some positive $\epsilon_0\simeq 0$ such that ${}^*n(\epsilon_0)\leqq N$, for if not, the internal set $\{\epsilon>0 \mid {}^*n(\epsilon)>N\}$ would properly contain the external set of positive infinitesimals, in which case there would exist a positive $\epsilon \not\simeq 0$ such that $*n(\epsilon) > N$, a contradiction. Now by transfer to $*B(G)$, the sentence "there exist $r_i \in *G$, $i = 1, 2, \cdots, *n(\epsilon_0)$, such that $*G = \bigcup r_i *E(\epsilon_0, f)$, $(1 \leqq i \leqq *n(\epsilon_0))$" is true. By setting $r_i = r_1$ for $*n(\epsilon_0) < i \leqq N$, and noting that $*E(\epsilon_0, f) \subset E(f)$, we have $\bigcup_{i=1}^{N} r_i E(f) = *G$ and the proof is complete.

It is easily seen that for any $f \in B(G)$, $E(f)$ is an invariant subgroup of $*G$ in analogy with the group of periods of a periodic function. Let $b_f G$ denote the quotient group $*G/E(f)$. Elements of $b_f G$ will be denoted by primed lower case letters. The function f has a natural extension f' defined on $b_f G$ by $f'(xE(f)) = {}^0(*f(x))$. This definition is clearly independent of the choice of the element $x \in *G$ to represent the coset $xE(f)$. The definition of f' is equivalent to the assertion that $f'(x') \simeq f(x)$ for all $x \in *G$, where $*f$ and $xE(f)$ have been abbreviated to f and x', respectively.

The sets $E(\epsilon, f)$, $\epsilon > 0$, form an open basis at the identity for a topology on G, and the sets $E'(\epsilon, f') = \{t' \in b_f G \mid$ for all $x', y' \in b_f G$, $|f(x't'y') - f(x'y')| < \epsilon\}$, $\epsilon > 0$, form an open basis at the identity for a topology on $b_f G$. It is easy to verify that G and $b_f G$ with these topologies are topological groups. The monads of the identity for the enlargements $*G$ and $*b_f G$ are, respectively, $E(f)$ and $E'(f') = \{t' \in *b_f G \mid$ for all $x', y' \in *b_f G, f'(x't'y') \simeq f'(x'y')\}$.

The following theorems describe the relationship between G and $b_f G$ where f is a fixed almost periodic function on G.

THEOREM 5. *Let ν denote the natural homomorphism of $*G$ onto $b_f G$. Then $\nu(G)$ is dense in $b_f G$.*

PROOF. Let $\epsilon > 0$ and suppose $x' \in b_f G$. Then $x' = xE(f)$ for some $x \in *G$, and since f is (r.)a.p., $*E(\epsilon/2, f)x \cap G \neq \varnothing$. Hence there exists $y \in G$ such that $yx^{-1} \in *E(\epsilon/2, f)$. Thus we have for all $z, w \in *G$, $|f(zyx^{-1}w) - f(zw)| < \epsilon/2$. By the definition of f', it follows that for all $z', w' \in b_f G$, $|f'(z'y'x'^{-1}w') - f'(z'w')| < \epsilon$. Thus every basic open neighborhood $E'(\epsilon, f')x'$ of $x' \in b_f G$ contains an element $y' = \nu(y) \in \nu(G)$. The proof is completed by using a similar argument based on the left almost periodicity of f to show that every basic open neighborhood $x'E'(\epsilon, f')$ of x' contains an element of $\nu(G)$.

The natural homomorphism ν: $*G \to b_f G$ is continuous in the sense that for any standard $\eta > 0$, $\nu(*E(\eta/2, f)) \subset E'(\eta, f')$. In fact, if $t \in *E(\eta/2, f)$, then $|f(xtz) - f(xz)| < \eta/2$ for all $x, z \in *G$, so that for all $x' = \nu(x)$ and $z' = \nu(z)$ in $b_f G$, we have

$$|f'(x't'z') - f'(x'z')| < \eta$$

because of the definition of f'. Thus $t' = \nu(t) \in E'(\eta, f')$. Since the restriction ν_0 of ν to G is also continuous, it follows by transfer to $^*G$ that $^*\nu_0$: $^*G \to {}^*b_fG$ has the property that for any nonstandard real number $\eta > 0$, $^*\nu_0(^*E(\eta/2, f)) \subset {}^*E(\eta, f)$. These continuity properties are needed for the following important lemma.

LEMMA. *For any $y \in {}^*G$, $\nu(y)$ and $^*\nu_0(y)$ are in the same monad in b_fG.*

PROOF. Let $\eta > 0$ be standard. Since f is a.p., there exists an element $u^{-1} \in G \cap {}^*E(\eta/2, f)y^{-1}$. By the continuity properties of ν and $^*\nu_0$, $\nu(u^{-1}y) \in E'(\eta, f')$ and $^*\nu_0(u^{-1}y) \in {}^*E'(\eta, f')$. Since the homomorphisms ν and $^*\nu_0$ agree on G, it follows that both $\nu(y)$ and $^*\nu_0(y)$ are elements of $\nu(u)^*E'(\eta, f')$ and hence that $^*\nu_0(y)\nu(y)^{-1} \in {}^*E'(2\eta, f')$. But this is true for any standard $\eta > 0$, so $^*\nu_0(y)\nu(y)^{-1} \in E'(f')$, i.e., $^*\nu_0(y)$ and $\nu(y)$ are in the same monad in $^*b_fG$.

THEOREM 6. *b_fG is compact.*

PROOF. The proof is based on the following nonstandard characterization of compactness: a topological space T is compact if and only if every element of $^*T$ is in the monad of some element of T [**4**, p. 93]. Suppose $x' \in {}^*b_fG$. Theorem 5 implies, by transfer to $^*B(G)$, that given any nonstandard real number $\epsilon > 0$, there exists $y \in {}^*G$ such that $^*E'(\epsilon, f')x'$ contains $^*\nu_0(y)$. Let $y' = \nu(y) \subset b_fG$. By the lemma, $^*\nu_0(y)$ and $\nu(y)$ are in the same monad in $^*b_fG$. But ϵ can be chosen to be infinitesimal. Hence x' and $^*\nu_0(y)$ are in the same monad in $^*b_fG$, because $^*E'(\epsilon, f') \subset E'(f')$. By transitivity, x' is in the monad of the point $\nu(y)$ in b_fG, and the proof is complete.

THEOREM 7. *b_fG is a Hausdorff space.*

PROOF. A topological space T is Hausdorff if and only if the monads of distinct points of T are disjoint [**4**, p. 92]. Suppose $x' \neq y'$ in b_fG. Since the monads $x'E'(f')$ and $y'E'(f')$ are cosets of the subgroup $E'(f')$ of $^*b_fG$, it suffices to show that there exists a point in one monad which is not in the other. Now $x' = \nu(x) = xE(f)$ and $y' = \nu(y) = yE(f)$ for some $x, y \in {}^*G$, so $xE(f) \cap yE(f) = \varnothing$ and $y^{-1}x \notin E(f)$. Thus for some $z, w \in {}^*G$ and some standard $\epsilon > 0$, $|f(zy^{-1}xw) - f(zw)| \geqq \epsilon$. But then by the definition of f', $|f'(z'y'^{-1}x'w') - f(z'w')| \geqq \epsilon/2$, which shows that $y'^{-1}x' \notin E'(f')$, i.e., $x' \notin y'E'(f')$. Of course, $x' \in x'E'(f')$, so the proof is complete.

THEOREM 8. *The set of all complex valued continuous functions on b_fG is isometrically isomorphic to the set of almost periodic functions g on G such that $E(g) \supset E(f)$.*

PROOF. If g is continuous on b_fG, then it is uniformly continuous, since b_fG is compact. It follows from the lemma that for any $z\in{}^*G$, $g'(\nu(z))\simeq g'({}^*\nu_0(z))$. Define $g\in B(G)$ by $g(u)=g'(u')=g'(\nu_0(u))$ for $u\in G$. Now for any x, $y\in{}^*G$ and $t\in E(f)$, $g(xty)=g'({}^*\nu_0(xty))$ and $g(xy)=g'({}^*\nu_0(xy))$ by transfer to ${}^*B(G)$, so

$$g(xty) \simeq g'(\nu(x)\nu(t)\nu(y)) = g'(\nu(x)\nu(y)) \simeq g(xy).$$

Here we have used the fact that $\nu(t)$ is the identity in ${}^*b_fG$. Hence $E(f)\subset E(g)$ and thus $E(g)$ must satisfy the conditions of Theorem 4. Therefore g is a.p.

Now suppose g is a.p. on G with $E(g)\supset E(f)$. Thus if $t\in E(f)\subset E(g)$, then $g(xty)\simeq g(xy)$ for all x, $y\in{}^*G$, which means that g satisfies the nonstandard criterion for uniform continuity with respect to the $E(\epsilon, f)$ topology (cf. [4, p. 111]). Hence for any $\eta>0$, there exists $\epsilon>0$ such that if $t\in E(\epsilon,f)$, then $|g(xty)-g(xy)|<\eta/2$ for all x, $y\in G$. By transfer, if $t\in{}^*E(\epsilon,f)$, then $|g(xty)-g(xy)|<\eta/2$ for all x, $y\in{}^*G$. Define g' on b_fG by $g'(x')={}^0(g(x))$ for $x'\in b_fG$. It is easy to check that g' is well defined. To demonstrate the continuity of g', let $\eta>0$, choose ϵ as above, and suppose $t'\in E'(\epsilon/2,f')\subset b_fG$. Writing $t'=\nu(t)$, it follows from $|f(xty)-f(xy)|\simeq|f'(x't'y')-f'(x'y')|<\epsilon/2$ that $t\in{}^*E(\epsilon, f)$. Now for any x', $y'\in b_fG$,

$$\begin{aligned} |g'(x't'y') - g'(x'y')| &= |{}^0(g(xty)) - {}^0(g(xy))| \\ &\simeq |g(xty) - g(xy)| < \eta/2. \end{aligned}$$

So $|g'(x't'y')-g'(x'y')|<\eta$ for all x', $y'\in b_fG$, which proves that g' is (uniformly) continuous on b_fG.

To show that the mappings $g\to g'$ and $g'\to g$ define a one-to-one correspondence, it suffices to verify that they are inverses of one another. Suppose first that g is a.p. Then $g'(x')={}^0(g(x))$ for $x'\in b_fG$, and g' maps to the function h defined by $h(x)=g'(x')$ for $x\in G$. But for $x\in G$, $g(x)={}^0(g(x))$, so $h=g$. Now let g' be continuous on b_fG. Then $g(x)=g'(\nu_0(x))$ for $x\in G$, and the image h' of g is defined by $h'(x')={}^0(g(x))$ for each $x'\in b_fG$. Thus $h'(x')={}^0(g'({}^*\nu_0(x)))$. By the lemma and the uniform continuity of g', $g'({}^*\nu_0(x))\simeq g'(\nu(x))=g'(x')$ for each $x\in{}^*G$, so $h'=g'$. It is easy to check that the correspondence $g\leftrightarrow g'$ is an isomorphism, and it is an isometry because

$$\sup_{x\in G} |g(x)| = \sup_{x\in{}^*G} |g(x)| = \sup_{x'\in b_fG} |g'(x')|.$$

REFERENCES

1. S. Bochner, *Beiträge zur Theorie der fastperiodischen Funktionen*, Math. Ann. **96** (1927), 119–147.

2. L. D. Kugler, "Non-standard analysis of almost periodic functions" *Applications of model theory to algebra, analysis and probability*, edited by W. A. J. Luxemburg, Holt, New York, 1969.

3. J. von Neumann, *Almost periodic functions in a group*. I, Trans. Amer. Math. Soc. **36** (1934), 445–492.

4. A. Robinson, *Non-standard analysis*, North-Holland, Amsterdam, 1966.

5. A. Weil, *L'integration dans les groupes topologiques et ses applications*, Actualités Sci. Indust., No. 869, Hermann, Paris, 1940.

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