LOCAL UNIFORM CONVEXITY OF DAY'S NORM ON $c_0(\Gamma)$

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1. **Introduction.** Let Γ be a nonempty set and let $c_0(\Gamma)$ denote the Banach space (supremum norm) of all real-valued functions x on Γ such that for each $\epsilon > 0$, $\{\gamma \in \Gamma : |x(\gamma)| \geqq \epsilon\}$ is finite. This space has received renewed interest because of a powerful mapping theorem of Lindenstrauss [4]: If E is a reflexive Banach space, then there exist a set Γ and a continuous one-to-one linear map T of E into $c_0(\Gamma)$. More generally, Amir and Lindenstrauss [1] have shown that if a Banach space E is the closed linear span of a weakly compact subset of E (i.e., if E is weakly compactly generated), then there exist such a set Γ and mapping T. The existence of such a map, together with Day's theorem [3] that $c_0(\Gamma)$ admits an equivalent strictly convex norm, makes it easy to show that every weakly compactly generated Banach space admits an equivalent strictly convex norm [1].

Consider, now, a stronger property than strict convexity; that of *local uniform convexity*:

(LUC) If $\|x_n\| = 1 = \|x\|$ and $\|x_n + x\| \to 2$, then $\|x_n - x\| \to 0$.

The main purpose of this note is to prove that a certain function on $c_0(\Gamma)$ defined by Day [3] is actually an equivalent (LUC) norm for $c_0(\Gamma)$. In §3 this fact is combined with the Lindenstrauss mapping theorem to obtain a new renorming result for reflexive Banach spaces.

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2. **Proof of the main result.** We first recall the norm on $c_0(\Gamma)$ defined by Day. If $x \in c_0(\Gamma)$, then x has countable support $E(x) = \{\alpha_k\}$, which can be enumerated so that $|x(\alpha_k)| \geqq |x(\alpha_{k+1})|$, $k = 1, 2, 3, \cdots$. Define $D: c_0(\Gamma) \to l_2(\Gamma)$ by

$$(Dx)(\gamma) = \frac{x(\alpha_k)}{2^k} \qquad \text{if } \gamma \in E(x)$$

$$= 0 \qquad \text{if } \gamma \notin E(x).$$

Although D is nonlinear, the function $p(x) = \|Dx\|_{l_2}$, $(x \in c_0(\Gamma))$ is a norm on $c_0(\Gamma)$. (It follows easily from the definition that $p(rx)$

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$=|r|p(x)$; we prove the triangle inequality below.) Since $|x(\alpha_1)| = \|x\|$, we see that $\|x\|/2 \leqq p(x) \leqq \|x\|/\sqrt{3}$, $(x \in c_0(\Gamma))$, so p is equivalent to the supremum norm on $c_0(\Gamma)$.

We next observe the following identity: If $s_1 \geqq s_2 \geqq \cdots \geqq 0$ and $t_1 \geqq t_2 \geqq \cdots \geqq 0$ and if β is any permutation of the positive integers, then

$$\sum_{k=1}^{\infty} s_k t_k - \sum_{k=1}^{\infty} s_k t_{\beta(k)} = \sum_{k=1}^{\infty} (s_k - s_{k+1}) \left[\sum_{i=1}^{k} t_i - \sum_{i=1}^{k} t_{\beta(i)} \right]$$

We can draw two conclusions from this:

(1) $\sum_k s_k t_k \geqq \sum_k s_k t_{\beta(k)}$, and

(2) For each integer m, $\sum_k s_k t_k - \sum_k s_k t_{\beta(k)} \geqq (s_m - s_{m+1})(t_m - t_{m+1})$ or β permutes $1, 2, \cdots, m$ onto itself.

Conclusion (1) follows from the fact that $\sum_1^k t_i \geqq \sum_1^k t_{\beta(i)}$, for each k, while (2) is immediate from the fact that if $\{\beta(i)\}_{i=1}^m \neq \{1, 2, \cdots, m\}$, then $t_1 + t_2 + \cdots + t_{m-1} + t_{m+1} \geqq \sum_{i=1}^m t_{\beta(i)}$.

It follows from (1) that if $x \in c_0(\Gamma)$, with $E(x) = \{\alpha_k\}$ (so that $\{|x(\alpha_k)|\}$ is nonincreasing), then

$$p(x)^2 \geqq \sum 4^{-k} |x(\beta_k)|^2 \tag{3}$$

for any permutation $\{\beta_k\}$ of $\{\alpha_k\}$. In fact, (3) holds for *any* sequence $\{\beta_k\}$ from Γ, since if $\beta_k \notin E(x)$, then we have introduced a zero term on the right side. This inequality allows us to prove the triangle inequality for p:

If x, y are in $c_0(\Gamma)$, let $E(x) = \{\alpha_k\}$, $E(y) = \{\beta_k\}$ and $E(x+y) = \{\gamma_k\}$. Then

$$\begin{aligned} p(x+y) &= \left(\sum 4^{-k}(x+y)(\gamma_k)^2\right)^{1/2} \\ &\leqq \left(\sum 4^{-k} x(\gamma_k)^2\right)^{1/2} + \left(\sum 4^{-k} y(\gamma_k)^2\right)^{1/2} \\ &\leqq \left(\sum 4^{-k} x(\alpha_k)^2\right)^{1/2} + \left(\sum 4^{-k} y(\beta_k)^2\right)^{1/2} = p(x) + p(y). \end{aligned}$$

To prove that p is (LUC), suppose that $p(x+x_n) \to 2p(x)$ and $p(x_n) \to p(x)$; we must show that $p(x-x_n) \to 0$. To this end, let $E(x) = \{\alpha_k\}$, $E(x_n) = \{\alpha_k^n\}$, and $E(x+x_n) = \{\beta_k^n\}$, and consider the difference

$$\begin{aligned} & 2p(x)^2 + 2p(x_n)^2 - p(x+x_n)^2 \\ &\quad = \sum 4^{-k}\left[2x(\alpha_k)^2 + 2x_n(\alpha_k^n)^2 - (x+x_n)(\beta_k^n)^2\right] \\ &\quad \geqq \sum 4^{-k}\left[2x(\beta_k^n)^2 + 2x_n(\beta_k^n)^2 - (x+x_n)(\beta_k^n)^2\right] \\ &\quad = \sum 4^{-k}\left[x(\beta_k^n) - x_n(\beta_k^n)\right]^2. \end{aligned} \tag{4}$$

Since the quantity in the first line converges to zero as $n\to\infty$, we see that

$$\lim_{n\to\infty}\left[x(\beta_k^n) - x_n(\beta_k^n)\right] = 0, \qquad k = 1, 2, 3, \cdots. \tag{5}$$

Suppose, now, that $p(x_n - x)\not\to 0$. Then there exists a subsequence (which we still denote by $\{x_n\}$) and $\epsilon>0$ such that $\|x_n - x\|\geqq\epsilon$ for each n. Let K be the largest integer which satisfies $|x(\alpha_K)|\geqq\epsilon/16$. Then $|x(\alpha_{K+1})|<\epsilon/16\leqq|x(\alpha_K)|$ and hence

$$0 < \delta = 2(4^{-K} - 4^{-K-1})(|x(\alpha_K)|^2 - |x(\alpha_{K+1})|^2).$$

If n is large enough that the first line of (4) is less than δ, then the second and third lines differ by less than δ, so that

$$\sum 4^{-k} 2x(\alpha_k)^2 - \sum 4^{-k} 2x(\beta_k^n)^2 < \delta.$$

From (2) it is readily seen that this is possible only if $\{\alpha_k\}_{k=1}^K = \{\beta_k^n\}_{k=1}^K$ for all large n. By choosing a subsequence, we can assume that $\beta_k^n = \beta_k$ for each n, $k=1, 2, \cdots, K$. From (5) it follows that $x_n(\beta_k)\to x(\beta_k)$, $k=1, 2, \cdots, K$, and since $\{\alpha_k\}_{k=1}^K = \{\beta_k\}_{k=1}^K$ we have $x_n\to x$ pointwise (hence uniformly) on the finite set $A = \{\alpha_k\}_{k=1}^K$.

For each n choose γ_n in Γ such that $|(x - x_n)(\gamma_n)| = \|x - x_n\|\geqq\epsilon$. By what we have just shown (and by the hypothesis $p(x_n)\to p(x)$) we can choose N large enough such that

$$\begin{aligned} &|(x - x_n)(\alpha)| < \epsilon && \text{if } \alpha\in A,\ n\geqq N \\ &x(\alpha)^2 - x_n(\alpha)^2 < \epsilon^2 4^{-K-4} && \text{if } \alpha\in A,\ n\geqq N \\ &p(x_n)^2 - p(x)^2 < \epsilon^2 4^{-K-4} && \text{if } n\geqq N. \end{aligned} \tag{6}$$

Suppose that $\gamma_n\notin A$. If we replace $E(x_n) = \{\alpha_k^n\}$ by a sequence which starts with $\alpha_1, \cdots, \alpha_K, \gamma_n$, then (3) implies that

$$\begin{aligned} p(x_n)^2 &= \sum \frac{x_n(\alpha_k^n)^2}{4^k} \\ &\geqq \sum_{k=1}^K \frac{x_n(\alpha_k)^2}{4^k} + \frac{x_n(\gamma_n)^2}{4^{K+1}}. \end{aligned} \tag{7}$$

Furthermore, since $|x(\alpha)|<\epsilon 4^{-2}$ if $\alpha\notin A$ and since $\sum_{K+1}^\infty 4^{-k} = (3\cdot 4^K)^{-1}$, we have

$$p(x)^2 < \sum_{k=1}^K \frac{x(\alpha_k)^2}{4^k} + \left(\frac{\epsilon}{4^2}\right)^2 \frac{1}{4^K\cdot 3}. \tag{8}$$

Using (7), then (8), and then (6), we obtain

$$\begin{aligned}\frac{x_n(\gamma_n)^2}{4^{K+1}} &\leqq p(x_n)^2 - p(x)^2 + p(x)^2 - \sum_{k=1}^{K} \frac{x_n(\alpha_k)^2}{4^k} \\ &< p(x_n)^2 - p(x)^2 + \sum_{k=1}^{K} \frac{x(\alpha_k)^2 - x_n(\alpha_k)^2}{4^k} + \frac{\epsilon^2}{4^4}\cdot\frac{1}{4^K\cdot 3} \\ &< \epsilon^2 4^{-K-3} \qquad \text{if } n \geqq N.\end{aligned}$$

Thus, if $n \geqq N$, we have (from (6)) $|(x-x_n)(\gamma_n)| < \epsilon$ if $\gamma_n \in A$ and $|(x-x_n)(\gamma_n)| \leqq |x(\gamma_n)| + |x_n(\gamma_n)| < 4^{-2}\epsilon + 4^{-1}\epsilon < \epsilon$ if $\gamma_n \notin A$, a contradiction which completes the proof.

3. **A renorming theorem.** It is an interesting open question whether every reflexive Banach space can be given an equivalent (LUC) norm. This is known [2, Proposition 2] to be equivalent to the problem of whether every such space can be given an equivalent Fréchet differentiable norm. (For related questions and results, see Asplund [2] and Lindenstrauss [5, §5].) The following result, however, is an easy consequence of the Lindenstrauss mapping theorem and the fact that Day's norm is (LUC).

PROPOSITION. *If E is a reflexive Banach space then E admits an equivalent norm $\|\cdot\cdot\|_1$ which is weakly locally uniformly convex, i.e., which satisfies*

(WLUC) *If* $\|x_n\|_1 = 1 = \|x\|_1$ *and* $\|x_n + x\|_1 \to 2$, *then* $x_n \to x$ *weakly.*

Before proving this, we prove a simple lemma.

LEMMA. *Suppose that E is a linear space with two norms $\|\cdot\cdot\|$ and $|\cdot\cdot|$, and that*

$$\|x\|_1 = (\|x\|^2 + |x|^2)^{1/2} \qquad (x \in E).$$

If $\{x_n\} \subset E$ and $x \in E$ are such that

$$(*) \qquad \|x_n\|_1 \to \|x\|_1 \quad \text{and} \quad \|x_n + x\|_1 \to 2\|x\|_1$$

then (*) *also holds for the norms $\|\cdot\cdot\|$ and $|\cdot\cdot|$.*

PROOF. Let $a_n = (\|x_n\| + \|x\|)^2 - \|x_n + x\|^2$, $b_n = (|x_n| + |x|)^2 - |x_n + x|^2$, $c_n = (\|x_n\| - \|x\|)^2$ and $d_n = (|x_n| - |x|)^2$. Each of these is nonnegative and

$$\begin{aligned}a_n + b_n + c_n + d_n = {} & 2(\|x_n\|^2 + \|x\|^2 + |x_n|^2 + |x|^2) \\ & - (\|x_n + x\|^2 + |x_n + x|^2).\end{aligned}$$

Our hypotheses imply that the right side converges to zero; hence each of the four sequences converges to zero.

We now prove the proposition. Let $T:E\to c_0(\Gamma)$ be the map obtained from Lindenstrauss' theorem and let p be Day's norm on $c_0(\Gamma)$. Denoting the norm on E by $\|\cdot\|$, define

$$\|x\|_1 = (\|x\|^2 + [p(Tx)]^2)^{1/2}, \qquad x \in E.$$

It is clear that $\|\cdot\cdot\|_1$ is an equivalent norm on E. Suppose that $\|x_n\|_1 = 1 = \|x\|_1$ and $\|x+x_n\|_1 \to 2$; we want to show that $x_n \to x$ weakly. By the lemma, we have $p(Tx_n)\to p(Tx)$ and $p(T(x+x_n)) = p(Tx+Tx_n) \to 2p(Tx)$. Since p is (LUC), we have $p(Tx_n - Tx)\to 0$. Now, since the sequence $\{x_n\}$ is bounded and E is reflexive, in order to show that $x_n \to x$ weakly it suffices to show that if (x_α) is a weakly convergent subnet of $\{x_n\}$, then $\lim x_\alpha = x$. But if $x_\alpha \to y$ weakly, then $Tx_\alpha \to Ty$ weakly; since $Tx_n \to Tx$, we have $Tx = Ty$. Since T is one-to-one, we have $x = y$, and the proof is complete.

Lindenstrauss [5] has shown that the space l_∞ of all bounded sequences does not admit an equivalent (WLUC) norm, although it clearly admits a linear one-to-one continuous map into c_0.

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