ON SMOOTH BOUNDED MANIFOLDS[1]

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A noncompact smooth manifold M is called a smooth bounded manifold if the following statement is true. There exist a compact subset $A \subset M$ and a smooth function g defined on $M-A$ such that g has no critical points on $M-A$ and such that $g(q_i) \to \infty$ for each sequence $q_i \in M$ without limit point on M.

We show that on each smooth bounded manifold there exists a smooth function f which has only a finite number of nondegenerate critical points on M and which has the property that $f(q_i) \to \infty$ for each sequence $q_i \in M$ without limit point on M.

Such a function will be called a critical finite function on M.

Together with the theorem of [**5**] it follows that to each strongly pseudoconvex Stein manifold with C^2-boundary (see [**4**, p. 262]) and with complex dimension m there exists a finite CW-complex $K \subset M$ of real dimension m such that K is a deformation retract of M. (K is the union of the finitely many descending bowls associated with some critical point of f, [**5**] see §2.)

In §§1 and 2 the manifold M is a smooth bounded manifold and the function g is given as above. M shall have a Riemannian metric.

1. **Motivation of "M is smooth bounded."** The theorem of this section shall give a motivation for the expression "M is a smooth bounded manifold". It will not be used in §§2 and 3.

Let C be a compact subset of M with $A \subset\subset C$, i.e. the closure of A is contained in C.

REMARK 1. There exists a number $e \in \boldsymbol{R}$ such that $B_b = \overline{B}_b = \{p \in M-C \mid g(p) = b\} \subset\subset M$ for $b \geqq e$.

PROOF. Let $C = \overline{C} \subset\subset M$ such that $A \subset\subset C$. The function g is defined on the boundary ∂C of C in M. Since C is compact it follows that $g(p) \leqq d$ on ∂C for some $d \in \boldsymbol{R}$. Let $e > d$. Then for each set B_b with $b \geqq e$ one has $B_b \cap \partial C = \emptyset$. Since $g(q_i) \to \infty$ for each sequence which has no limit point in M it follows that $B_b = \overline{B}_b \subset\subset M$.

In the following, C shall always be the subset of M of Remark 1. Let e and B_b be as in Remark 1.

REMARK 2. Let $b > e$ and $\psi(q)$ be the maximal orthogonal trajectory of g through $q \in M-C$. Then there exists a parametrization

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$\{\psi_t(p) \mid t \in \boldsymbol{R}\}$ of $\psi(p)$ with $p \in B_b$ such that $g(\psi_t(p)) = t + b$ if $t + b \geqq e$.

PROOF. Since g has no critical points in $\{p \in M - C \mid g(p) \geqq e\}$ one can use the theorem in [7, p. 13] to obtain in each compact set $\{p \in M - C \mid a_i \leqq g(p) \leqq a_{i+1}\}$ the result of Remark 2, where $a_1 < a_2 < a_3 < \cdots$ with $\lim_{i \to \infty} a_i = \infty$. These parametrizations yield then the parametrization of Remark 2.

There exists a smooth function $g'(q)$ on $M - C$ such that $g'(q) = 0$ if $g(q) \geqq d$ and $g'(q) = 1$ if $g(q) \leqq b$ and $0 \leqq g'(q) \leqq 1$ on $M - C$ and such that the functions g' and g have the same level surfaces on $\{q \in M - C \mid b < g(q) < d\}$ (see [5, Remark 1]).

Below we will denote this function by $g_{b,d,g}(q)$ for $q \in M - C$.

THEOREM 1. *Let M be a smooth bounded manifold. There then exists a compactification (see [2, p. 85]) $M' = M \cup B$ of M such that B is a smooth manifold of dimension $n-1$ and M' is a smooth manifold with boundary B.*

PROOF. We show that there exists a diffeomorphism which maps M on a proper subset $D \subset M$ such that the boundary B' of D in M is an $(n-1)$-dimensional manifold. Let C, e and the parametrization of $\psi(p)$ be as above. Let $b > e$ and $d - b < 1$ and $g'(q) = 1 - g_{b,d,g}(q)$ if $q \in M - C$. Define $h(q) = h(\psi_r(p)) = \psi_t(p)$ with $p \in B_b$ and $0 \leqq r < 1$ and $t = r/(1 - g'(\psi_r(p))r)$. Let $h(q) = q$ if $g(q) \leqq b$ or $q \in C$. Then h is a diffeomorphism of $D = \{q \in M \mid q \in C$ or $g(q) < b + 1\}$ onto M. Since $B' = \{q \in M - C \mid g(q) = b + 1\}$ is an $(n-1)$-dimensional smooth manifold and $D \cup B'$ is a smooth manifold with boundary B', Theorem 1 is true.

2. **Critical finite functions on a smooth bounded manifold.** Let $M \subset \boldsymbol{R}^k$ be differentiably embedded in $\boldsymbol{R}^k$. Let $x \in \boldsymbol{R}^k$ be a fixed point and $L_x(p) = |p - x|^2$ the euclidian distance between p and x for $p \in M$. The function L_x is smooth on M.

The following two propositions are from [7, p. 36f].

PROPOSITION 1. *For almost all $x \in \boldsymbol{R}^k$ (all but a set of measure 0) the function L_x has no degenerate critical points.*

PROPOSITION 2. *Let h be a smooth function on M such that $\{p \in M \mid h(p) \leqq c\} \subset\subset M$ for each $c \in \boldsymbol{R}$. Let $K \subset M$ be a compact set. Then h can be uniformly approximated on K by a smooth function f on M which has no degenerate critical points. Furthermore f can be chosen such that the* ith *derivative of f on the compact set K uniformly approximates the corresponding derivative of h for $i = 1, 2$. One has $f = c \cdot L_x + d$ for some $x \in \boldsymbol{R}^k$ and some constants $c > 0$ and d.*

THEOREM 2. *Let M be a smooth bounded manifold. Then there is a critical finite function f' on M.*

PROOF. Let $a>a'\geqq e$ and $g_1(p)=g_{a',a,g}(p)$ for $p\in M-C$ (see Remark 1). Define $g^+(p)=(1-g_1(p))g(p)+g_1(p)a'$ if $g(p)\geqq a'$, $p\notin C$ and $g^+(p)=a'$ if $p\in C$ or $g(p)<a'$. Let $\Delta g(p)$ be the gradient of g. Denote by $\Delta g(f(p))$ the directional derivative of a function f at $p\in M$ along $\Delta g(p)$. The function g^+ is smooth, and $\Delta g(g^+(p)) =(1-g_1(p))\Delta g(g(p))+(a'-g(p))\Delta g(g_1(p))>0$ if $g_1(p)<1$. Let $b>a$ and $K=C\cup\{p\in M-C\mid g(p)\leqq b\}$. Replace h,f by g^+,f^+ respectively; then Proposition 2 yields the existence of a function f^+ which approximates g^+ on K. One has $f^+(p)<b'$ for some $b'\in\boldsymbol{R}$ and $p\in\{q\in M\mid g^+(q)\leqq b\}$. Let $g_2(p)=g_{a,b,g}(p)$ if $p\in M-C$. Since $g^+=g$ on $E=\{p\in M-C\mid a\leqq g(p)\leqq b\}$ it follows for the inner product $(\Delta g^+(p),\ \Delta g^+(p))\geqq c'>0$ if $p\in E$. Hence, since the gradient $\Delta f^+(p)$ approximates $\Delta g^+(p)$ (Proposition 2), $(\Delta g^+(p),\Delta f^+(p))>0$ on E.

Define $f'(p)=(1-g_2(p))(g^+(p)+b'-a)+g_2(p)f^+(p)$ if $p\in M-C$ and $f'(p)=f^+(p)$ if $p\in C$. One has $\Delta g(f'(p))=(1-g_2(p))\Delta g(g^+(p)) +g_2(p)\Delta g(f^+(p))+(-g^+(p)-b'+a+f^+(p))\Delta g(g_2(p))>0$ if $p\in E$. The function f' is a critical finite function on M.

COROLLARY 1. *The function f' of Theorem 2 coincides with some function $c\cdot L_x+d$ with $c>0$ on a subset of M which contains all critical points of f'.*

PROOF. The construction of f^+ according to Proposition 2 is chosen such that f^+ equals some function $c\cdot L_x+d$ on M with $c>0$. Then $f'=c\cdot L_x+d$ on the set $C\cup\{p\in M-C\mid g(p)\leqq a\}$.

By [8, pp. 353 and 383] there exists a critical finite function f on M with critical points $p_1,\ p_2,\ \cdots,\ p_r$ on M such that $f(p_1)<f(p_2) <\cdots<f(p_r)$ and the index λ_i of f at p_i is less than dim M.

Let K be the union of the descending bowls E_i associated with some critical point p_i of f. More explicitly, let p be a noncritical point of f; then $\phi(p)$ denotes the maximal orthogonal trajectory of f through p, or if p is a critical point of f then $\phi(p)=\{p\}$. The set E_i is defined as follows:

$$E_i=\{p\in M\mid p_i\in\overline{\phi(p)},\ f(p)\leqq f(p_i)\}.$$

Let dim $K=\max_{i=1,\cdots,r}\lambda_i$.

COROLLARY 2. *K is a deformation retract of M and* dim $K\leqq$ dim $M-1$.

PROOF. The corollary of [5] yields the fact that K is a deformation retract of M. Since $\lambda_i\leqq$dim $M-1$ one has dim $K\leqq$dim $M-1$.

3. Strongly pseudoconvex Stein manifolds with C^2-boundary are smooth bounded.

PROPOSITION 3. *A strongly pseudoconvex Stein manifold M with C^2-boundary is smooth bounded.*

PROOF. From [3] and [4, p. 263] one has the existence of a strongly plurisubharmonic smooth function h defined in a neighborhood U of the boundary of M such that $M \cap U = \{p \in M \mid h(p) < 0\}$ and such that the gradient $\Delta h(p) \neq 0$ in $M \cap U$. Let $g(p) = -\log|h(p)|$. Since $A = M - U$ is compact it follows that M is a smooth bounded manifold.

THEOREM 3. *Let M be a strongly pseudoconvex Stein manifold with C^2-boundary of complex dimension m. There is a finite* CW-*complex $K \subset M$ with real dimension $\leqq m$ such that K is a deformation retract of M.*

PROOF. From Proposition 3 it follows that one can apply Theorem 2. Let f' be the critical finite function of Theorem 2. From [1] and Corollary 1 it follows that for each critical point of f' the index is less than or equal to m. Corollary 2 yields then the statement of Theorem 3.

REFERENCES

1. A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. of Math (2) **69** (1959), 713–717.

2. C. Constantinescu and A. Cornea, *Ideale Raender Riemannscher Flaechen*, Ergebnisse der Math. und ihrer Grenzgebiete, Springer, Berlin, 1963.

3. H. Grauert, *Ueber Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.

4. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

5. G. Kalmbach, *Deformation retracts and weak deformation retracts of noncompact manifolds*, Proc. Amer. Math. Soc. **20** (1969), 539–544.

6. ———, *Bowl functions on noncompact manifolds*, Mimeographed Notes, Univ. of Illinois, Urbana, 1967.

7. J. Milnor, *Morse theory*, Ann. of Math. Studies, No. 51, Princeton Univ. Press, Princeton, N. J., 1963.

8. M. Morse, *The existence of polar non-degenerate functions on differentiable manifolds*, Ann. of Math. (2) **71** (1960), 352–383.

9. ———, "Bowls of a nondegenerate function on a compact differentiable manifold" in *Differential and combinatorial topology*, Princeton Univ. Press, Princeton, N. J., 1965, pp. 81–103.

10. J. H. C. Whitehead, *Combinatorial homotopy*. I, Bull. Amer. Math. Soc. **55** (1949), 213–245.

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