ON SMOOTH BOUNDED MANIFOLDS

GUDRUN KALMBACH

A noncompact smooth manifold \( M \) is called a smooth bounded manifold if the following statement is true. There exist a compact subset \( A \subset M \) and a smooth function \( g \) defined on \( M - A \) such that \( g \) has no critical points on \( M - A \) and such that \( g(q_i) \to \infty \) for each sequence \( q_i \in M \) without limit point on \( M \).

We show that on each smooth bounded manifold there exists a smooth function \( f \) which has only a finite number of nondegenerate critical points on \( M \) and which has the property that \( f(q_i) \to \infty \) for each sequence \( q_i \in M \) without limit point on \( M \).

Such a function will be called a critical finite function on \( M \).

Together with the theorem of [5] it follows that to each strongly pseudoconvex Stein manifold with \( C^2 \)-boundary (see [4, p. 262]) and with complex dimension \( m \) there exists a finite CW-complex \( K \subset M \) of real dimension \( m \) such that \( K \) is a deformation retract of \( M \). (\( K \) is the union of the finitely many descending bowls associated with some critical point of \( f \), [5] see §2.)

In §§1 and 2 the manifold \( M \) is a smooth bounded manifold and the function \( g \) is given as above. \( M \) shall have a Riemannian metric.

1. Motivation of “\( M \) is smooth bounded.” The theorem of this section shall give a motivation for the expression “\( M \) is a smooth bounded manifold”. It will not be used in §§2 and 3.

Let \( C \) be a compact subset of \( M \) with \( A \subset C \), i.e. the closure of \( A \) is contained in \( C \).

Remark 1. There exists a number \( e \in \mathbb{R} \) such that \( B_b = \overline{B}_b = \{ p \in M - A \mid g(p) = b \} \subset M \) for \( b \geq e \).

Proof. Let \( C = \overline{C} \subset C \) such that \( A \subset C \). The function \( g \) is defined on the boundary \( \partial C \) of \( C \) in \( M \). Since \( C \) is compact it follows that \( g(p) \leq d \) on \( \partial C \) for some \( d \in \mathbb{R} \). Let \( e > d \). Then for each set \( B_b \) with \( b \geq e \) one has \( B_b \cap \partial C = \emptyset \). Since \( g(q_i) \to \infty \) for each sequence which has no limit point in \( M \) it follows that \( B_b = \overline{B}_b \subset M \).

In the following, \( C \) shall always be the subset of \( M \) of Remark 1. Let \( e \) and \( B_b \) be as in Remark 1.

Remark 2. Let \( b > e \) and \( \psi(g) \) be the maximal orthogonal trajectory of \( g \) through \( q \in M - C \). Then there exists a parametrization
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\[ \{ \psi_t(p) \mid t \in \mathbb{R} \} \] of \( \psi(p) \) with \( p \in B_b \) such that \( g(\psi_t(p)) = t + b \) if \( t + b \geq e \).

**Proof.** Since \( g \) has no critical points in \( \{ p \in M - C \mid g(p) \geq e \} \), one can use the theorem in [7, p. 13] to obtain in each compact set \( \{ p \in M - C \mid a_i \leq g(p) \leq a_{i+1} \} \) the result of Remark 2, where \( a_1 < a_2 < a_3 < \cdots \) with \( \lim_{i \to \infty} a_i = \infty \). These parametrizations yield then the parametrization of Remark 2.

There exists a smooth function \( g'(q) \) on \( M - C \) such that \( g'(q) = 0 \) if \( g(q) \geq d \) and \( g'(q) = 1 \) if \( g(q) \leq b \) and \( 0 \leq g'(q) \leq 1 \) on \( M - C \) and such that the functions \( g' \) and \( g \) have the same level surfaces on \( \{ q \in M - C \mid b < g(q) < d \} \) (see [5, Remark 1]).

Below we will denote this function by \( g_b,d,q(q) \) for \( q \in M - C \).

**Theorem 1.** Let \( M \) be a smooth bounded manifold. There then exists a compactification (see [2, p. 85]) \( M' = M \cup B \) of \( M \) such that \( B \) is a smooth manifold of dimension \( n - 1 \) and \( M' \) is a smooth manifold with boundary \( B \).

**Proof.** We show that there exists a diffeomorphism which maps \( M \) on a proper subset \( D \subset M \) such that the boundary \( B' \) of \( D \) in \( M \) is an \( (n - 1) \)-dimensional manifold. Let \( C \), \( e \) and the parametrization of \( \psi(p) \) be as above. Let \( b > e \) and \( d - b < 1 \) and \( g'(q) = 1 - g_b,d,q(q) \) if \( q \in M - C \). Define \( h(q) = h(\psi_r(p)) = \psi_r(p) \) with \( p \in B_b \) and \( 0 \leq r < 1 \) and \( t = r/(1 - g'(\psi_r(p))) \). Let \( h(q) = q \) if \( g(q) \leq b \) or \( q \in C \). Then \( h \) is a diffeomorphism of \( D = \{ q \in M \mid q \in C \) or \( g(q) < b + 1 \} \) onto \( M \). Since \( B' = \{ q \in M - C \mid g(q) = b + 1 \} \) is an \( (n - 1) \)-dimensional smooth manifold and \( D \cup B' \) is a smooth manifold with boundary \( B' \), Theorem 1 is true.

2. Critical finite functions on a smooth bounded manifold. Let \( M \subset \mathbb{R}^k \) be differentiably embedded in \( \mathbb{R}^k \). Let \( x \in \mathbb{R}^k \) be a fixed point and \( L_x(p) = |p - x|^2 \) the euclidian distance between \( p \) and \( x \) for \( p \in M \). The function \( L_x \) is smooth on \( M \).

The following two propositions are from [7, p. 36f].

**Proposition 1.** For almost all \( x \in \mathbb{R}^k \) (all but a set of measure 0) the function \( L_x \) has no degenerate critical points.

**Proposition 2.** Let \( h \) be a smooth function on \( M \) such that \( \{ p \in M \mid h(p) \leq c \} \subset M \) for each \( c \in \mathbb{R} \). Let \( K \subset M \) be a compact set. Then \( h \) can be uniformly approximated on \( K \) by a smooth function \( f \) on \( M \) which has no degenerate critical points. Furthermore \( f \) can be chosen such that the \( i \)th derivative of \( f \) on the compact set \( K \) uniformly approximates the corresponding derivative of \( h \) for \( i = 1, 2 \). One has \( f = c \cdot L_x + d \) for some \( x \in \mathbb{R}^k \) and some constants \( c > 0 \) and \( d \).
Theorem 2. Let \( M \) be a smooth bounded manifold. Then there is a critical finite function \( f' \) on \( M \).

Proof. Let \( a > a' \geq c \) and \( g_{i}(p) = g_{i, a, a'}(p) \) for \( p \in M - C \) (see Remark 1). Define \( g^{+}(p) = (1 - g_{i}(p))g(p) + g_{i}(p)a' \) if \( g(p) \geq a' \), \( p \in C \) and \( g^{+}(p) = a' \) if \( p \in C \) or \( g(p) < a' \). Let \( \Delta g(p) \) be the gradient of \( g \). Denote by \( \Delta g(f(p)) \) the directional derivative of a function \( f \) at \( p \in M \) along \( \Delta g(p) \). The function \( g^{+} \) is smooth, and \( \Delta g(g^{+}(p)) = (1 - g_{i}(p))\Delta g(g(p)) + (a' - g(p))\Delta g(g_{i}(p)) > 0 \) if \( g_{i}(p) < 1 \). Let \( b > a \) and \( K = \bigcup \{ p \in M - C \mid g(p) \leq b \} \). Replace \( h, f \) by \( g^{+}, f^{+} \) respectively; then Proposition 2 yields the existence of a function \( f^{+} \) which approximates \( g^{+} \) on \( K \). One has \( f^{+}(p) < b' \) for some \( b' \in \mathbb{R} \) and \( p \in \{ g \in M \mid g^{+}(p) \leq b \} \). Let \( g_{2}(p) = g_{a, b, a'}(p) \) if \( p \in M - C \). Since \( g^{+} = g \) on \( \{ p \in M - C \mid a \leq g(p) \leq b \} \) it follows for the inner product \( \langle \Delta g(p), \Delta g^{+}(p) \rangle \geq c' > 0 \) if \( p \in E \). Hence, since the gradient \( \Delta f^{+}(p) \) approximates \( \Delta g^{+}(p) \) (Proposition 2), \( \langle \Delta g^{+}(p), \Delta f^{+}(p) \rangle > 0 \) on \( E \).

Define \( f'(p) = (1 - g_{2}(p))(g^{+}(p) + b' - a) + g_{2}(p)f^{+}(p) \) if \( p \in M - C \) and \( f'(p) = f^{+}(p) \) if \( p \in C \). One has \( \Delta g(f'(p)) = (1 - g_{2}(p))\Delta g(g^{+}(p)) + g_{2}(p)\Delta g(f^{+}(p)) + (g^{+}(p) - b' + a + f^{+}(p))\Delta g(g_{2}(p)) > 0 \) if \( p \in E \). The function \( f' \) is a critical finite function on \( M \).

Corollary 1. The function \( f' \) of Theorem 2 coincides with some function \( c \cdot L_{a} + d \) with \( c > 0 \) on a subset of \( M \) which contains all critical points of \( f' \).

Proof. The construction of \( f^{+} \) according to Proposition 2 is chosen such that \( f^{+} \) equals some function \( c \cdot L_{a} + d \) on \( M \) with \( c > 0 \). Then \( f' = c \cdot L_{a} + d \) on the set \( C \cup \{ p \in M - C \mid g(p) \leq a \} \).

By [8, pp. 353 and 383] there exists a critical finite function \( f \) on \( M \) with critical points \( p_1, p_2, \ldots, p_r \) on \( M \) such that \( f(p_1) < f(p_2) < \cdots < f(p_r) \) and the index \( \lambda_i \) of \( f \) at \( p_i \) is less than \( \dim M \).

Let \( K \) be the union of the descending bowls \( E_i \) associated with some critical point \( p_i \) of \( f \). More explicitly, let \( p \) be a noncritical point of \( f \); then \( \phi(p) \) denotes the maximal orthogonal trajectory of \( f \) through \( p \), or if \( p \) is a critical point of \( f \) then \( \phi(p) = \{ p \} \). The set \( E_i \) is defined as follows:

\[
E_i = \{ p \in M \mid p_i \in \overline{\phi(p)}, f(p) \leq f(p_i) \}.
\]

Let \( \dim K = \max_{i=1, \ldots, r} \lambda_i \).

Corollary 2. \( K \) is a deformation retract of \( M \) and \( \dim K \leq \dim M - 1 \).

Proof. The corollary of [5] yields the fact that \( K \) is a deformation retract of \( M \). Since \( \lambda_i \leq \dim M - 1 \) one has \( \dim K \leq \dim M - 1 \).
3. Strongly pseudoconvex Stein manifolds with $C^2$-boundary are smooth bounded.

**Proposition 3.** A strongly pseudoconvex Stein manifold $M$ with $C^2$-boundary is smooth bounded.

**Proof.** From [3] and [4, p. 263] one has the existence of a strongly plurisubharmonic smooth function $h$ defined in a neighborhood $U$ of the boundary of $M$ such that $M \cap U = \{ p \in M \mid h(p) < 0 \}$ and such that the gradient $\Delta h(p) \neq 0$ in $M \cap U$. Let $g(p) = -\log |h(p)|$. Since $A = M - U$ is compact it follows that $M$ is a smooth bounded manifold.

**Theorem 3.** Let $M$ be a strongly pseudoconvex Stein manifold with $C^2$-boundary of complex dimension $m$. There is a finite CW-complex $K \subset M$ with real dimension $\leq m$ such that $K$ is a deformation retract of $M$.

**Proof.** From Proposition 3 it follows that one can apply Theorem 2. Let $f'$ be the critical finite function of Theorem 2. From [1] and Corollary 1 it follows that for each critical point of $f'$ the index is less than or equal to $m$. Corollary 2 yields then the statement of Theorem 3.

**References**

6. ———, *Bowl functions on noncompact manifolds*, Mimeographed Notes, Univ. of Illinois, Urbana, 1967.

**University of Illinois**