THE INVARIANT DISTANCE IN THE THEORY OF PSEUDO-CONFORMAL TRANSFORMATIONS AND THE LU QIKENG CONJECTURE

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Preliminary remarks. In the theory of pseudoconformal transformations the Bergman function plays an important role. It gives rise to numerous pseudoconformal invariants, for example, the Bergman metric form and the invariants $J^{(n)}$ (see [1], [2]). For a given domain $D\subset C^n$, the Bergman function $K_D(z, \bar{t})$ is uniquely determined as a holomorphic function of $2n$ complex variables, defined on the Cartesian product[2] $D\times\bar{D}$. Therefore, a study of the relations between the geometrical properties of the domain D and the function-theoretical properties of its Bergman function is of considerable interest. To this class belong the investigations of the behavior of $K_D(z, \bar{z})$ at the boundary (cf. [3], [7], [8], [11]).

In the present paper we shall be concerned with another property of the Bergman function, namely, with the solvability of the equation $K_D(z, \bar{t})=0$ in $D\times\bar{D}$. This property is invariant under pseudoconformal transformations. We will show also its relation with a certain property of the invariant distance in the domain D; the latter notion will be defined below.

1. The imbedding χ. Consider a bounded domain D in space of n-complex variables. The functions holomorphic in D and square integrable with respect to the Lebesgue measure in D, with a scalar product

$$\int_D f(z)\overline{g(z)}d\omega_z = (f, g),$$

form a complete Hilbert space, usually denoted by $L^2H(D)$. For a given point $t\in D$ denote by $\chi(t)$ the unique element in $L^2H(D)$ with the property $f(t)=(f, \chi(t))$, $f\in L^2H(D)$. The function $K_D(z, \bar{t}) =\chi(t)$ is called the Bergman function of the domain D. It is a holomorphic function of variables z, $\bar{t}$ in the Cartesian product

Presented to the Society February 1, 1968; received by the editors June 3, 1968.

[1] This work was partially supported by grant AF 1047/66 and contract AEC 326-P22.

[2] Here $\bar{D}$ denotes the set of all points whose coordinates are conjugate to the coordinates of a point in D.

$D\times\overline{D}$. The mapping χ: $D\to L^2H(D)$ defines an antianalytic imbedding of D into the Hilbert space.

2. **Pseudoconformal mappings and canonical isometry.** Let τ be a pseudoconformal mapping from a bounded domain $B\subset C^n$ onto D, that is, $z=\tau(u)$ is a one-to-one holomorphic mapping with nonvanishing determinant $\partial\tau/\partial u$, $u\in B$. This mapping induces a canonical linear isometry T: $L^2H(D)\to L^2H(B)$ given by

$$(Tf)(u) = f(\tau(u))\partial\tau/\partial u, \qquad f\in L^2H(D).$$

We note that T is onto and that

$$\chi_B(w) = \overline{(\partial\tau/\partial w)}T(\chi_D(t)), \qquad t = \tau(w).$$

If we denote

$$Z_D(t) = \left(\frac{e^{i\phi}\chi_D(t)}{\|\chi_D(t)\|}: 0 \leqq \phi \leqq 2\pi\right),$$

we obtain the following conclusion:

LEMMA 1. *The canonical isometry T transforms the circle $Z_D(t)$ onto the circle $Z_B(w)$.*

3. **The invariant distance.** Since T is an isometry, we conclude that the quantity

$$\rho(t_1, t_2) = \operatorname{dist}(Z_D(t_1), Z_D(t_2)) \tag{1}$$

is invariant under pseudoconformal transformation. Furthermore, we have

THEOREM 1. *Equation* (1) *defines a distance function in the domain D, i.e.,*

(i) $\rho(t_1, t_2)\geqq 0$, $\rho(t_1, t)_2=0\equiv t_1=t_2$, $t_1, t_2\in D$,

(ii) $\rho(t_1, t_2)=\rho(t_2, t_1)$,

(iii) $\rho(t_1, t_2)\leqq\rho(t_1, t)+\rho(t, t_2)$, $t\in D$.

PROOF. Property (i) follows from the fact that in a bounded domain D the elements $\chi(t_1)$ and $\chi(t_2)$ are linearly independent for $t_1\neq t_2$. One verifies immediately property (ii), and (iii) can be concluded from the identity

$$\min_{\phi,\psi}\|e^{i\phi}f - e^{i\psi}g\| = \min_{\psi}\|f - e^{i\psi}g\|, \qquad f, g \in L^2H(D).$$

The quantity $\rho(t_1, t_2)$ was mentioned previously in a paper by

Kobayashi [9, p. 280]. However, in the present form it can easily be expressed by the Bergman function, namely,

$$\rho(t_1, t_2) = \min_{\phi,\psi} \left\| \frac{e^{i\phi}\chi(t_1)}{\|\chi(t_1)\|} - \frac{e^{i\psi}\chi(t_2)}{\|\chi(t_2)\|} \right\| = \left(2 - 2\left(\frac{K(t_1, \bar{t}_2)K(t_2, \bar{t}_1)}{K(t_1, \bar{t}_1)K(t_2, \bar{t}_2)}\right)^{1/2}\right)^{1/2}$$
$$= \left(2 - 2\left|\left(\frac{\chi(t_1)}{\|\chi(t_1)\|}, \frac{\chi(t_2)}{\|\chi(t_2)\|}\right)\right|\right)^{1/2}.$$

In case of the unit disc Δ, where $K_\Delta(t_1, \bar{t}_2) = 1/\pi(1 - t_1\bar{t}_2)^2$ (see [5]), we obtain

$$\rho_\Delta(t_1, t_2) = 2^{1/2} \frac{|t_1 - t_2|}{|1 - t_1 t_2|}.$$

This shows that the Schwarz-Pick lemma can easily be stated in terms of the invariant distance. The second differential of ρ^2 yields the Bergman metric form

$$\text{(2)} \quad d^2\rho^2(p, t)\big|_{t=p} = \frac{2}{K(p, \bar{p})^2} \sum_{\alpha,\beta=1}^{n} \left(K(p, \bar{p}) \frac{\partial^2 K(p, \bar{p})}{\partial p^\alpha \partial \bar{p}^\beta} - \frac{\partial K(p, \bar{p})}{\partial p^\alpha} \frac{\partial K(p, \bar{p})}{\partial \bar{p}^\beta}\right) dt^\alpha d\bar{t}^\beta,$$

$$p = (p^1, p^2, \cdots, p^n), \; t = (t^1, t^2, \cdots, t^n) \in D.$$

4. Critical points of the invariant distance. In the following we will employ tensor notation. Italic indices shall run from 1 to $2n$, Greek indices from 1 to n, and barring an index means changing its value by n (see [13, p. 117]). We can now write the form (2) as

$$d^2\rho^2(p, t)\big|_{t=p} = T_{i,j} dt^i dt^j,$$

where

$$T_{\alpha,\bar{\beta}} = \frac{1}{K^2}\left(K \frac{\partial^2 K}{\partial p^\alpha \partial p^{\bar{\beta}}} - \frac{\partial K}{\partial p^\alpha} \frac{\partial K}{\partial p^{\bar{\beta}}}\right),$$

$$T_{\alpha,\beta} = T_{\bar{\alpha},\bar{\beta}} = 0, \qquad T_{\alpha,\bar{\beta}} = T_{\bar{\beta},\alpha} = \overline{T_{\bar{\alpha},\beta}}.$$

Let t be a fixed point of the domain D. Let us denote by ν^α the (contravariant) representative coordinates in D. The functions ν^α, meromorphic in D, map D onto the Bergman representative domain with respect to the point t (see Bergman [4] and Lu Qi-Keng [10, pp. 284 and 293]). We introduce the following definition.

Definition. The system of meromorphic functions

$$\text{(3)} \qquad \mu_{\bar{\beta}} = T_{\alpha,\bar{\beta}}\nu^\alpha$$

will be called the covariant representative coordinates (with respect to t) of the domain D.

The formal computation shows that

$$\mu_{\bar{\beta}} = \frac{\partial K(p,\bar{t})/\partial \bar{t}}{K(p,\bar{t})} - \frac{\partial K(t,\bar{t})/\partial \bar{t}}{K(t,\bar{t})}.$$

We are now in the position to state

THEOREM 2. *In order for the point $t\in D$ to be a critical point of the invariant distance $\rho(p, t)$ (considered as a function of the second variable), it is necessary and sufficient that one of the following conditions holds:*

(*a*) $K(p, \bar{t}) = 0$.

(*b*) *In the mapping onto the Bergman representative domain with respect to the point t, both points p and t correspond to the origin.*

PROOF. The critical points of $\rho(p, t)$ are exactly those of the expression

$$Q = K(p,\bar{t})K(t,\bar{p})/K(t,\bar{t}). \tag{4}$$

The first differential of (4) is equal to $Q(\bar{\mu}_{\bar{\beta}}dt^{\beta}+\mu_{\bar{\beta}}d\bar{t}^{\beta})$ and vanishes either with (a), or when $\mu_{\bar{\beta}}=0$. However, since the transformation (3) is linear, the latter occurs when all contravariant representative coordinates vanish and this in turn means (b).

Lu Qi-Keng pointed out in [10, p. 293], that many concrete examples justify an assumption that (a) can never occur, although no proof of this is known. We shall call this conjecture the Lu Qi-Keng conjecture and introduce the following definition.

DEFINITION. A domain $D\subset C^n$ will be called a Lu Qi-Keng domain if the equation $K_D(p, \bar{t})=0$ has no solution in $D\times\bar{D}$.

THEOREM 3. *The class of Lu Qi-Keng domains is closed under*

(1) *pseudoconformal transformations,*

(2) *forming a Cartesian product,*

(3) *approximation from inside. That is to say, if $D_m\subset D$ is a sequence of Lu Qi-Keng domains and* $\lim D_m = D$ *is bounded, then D is a Lu Qi-Keng domain.*

PROOF. (1) and (2) are immediate consequences of the corresponding formulas for the Bergman function; see [5, p. 136], and [6, p. 368], respectively. (3) is a conclusion from the recent result by Ramadanov, who proved that under conditions in (3) the sequence $K_{D_n}(p, \bar{t})$ converges locally uniformly to $K_D(p, \bar{t})$ on $D\times\bar{D}$; see

[**12**, pp. 759–762]. (The proof in [**12**] is formulated in a slightly less general context; however, with only minor changes it applies to the general case which we are concerned with.) Our argument is by contradiction. Suppose $K_D(p, \bar{t})$ has zeros in $D\times\overline{D}$. Then by Hurwitz' theorem for sufficiently large m, $K_{D_m}(p, \bar{t})$ has zeros in $D\times\overline{D}$, which is in contradiction with the assumption that D_m is a Lu Qi-Keng domain.

However, the Lu Qi-Keng conjecture is in general not true for multiply-connected domains. We conclude our considerations with the following counterexample.

EXAMPLE. Consider in C^1 an annulus $R=\{z: 0<r<|z|<1\}$. For $r<e^{-2}$, R is not a Lu Qi-Keng domain.

PROOF. We begin with a series which expresses the Bergman function for the annulus; see [**5**, p. 9]. It can also be written in the form

$$K(z, \bar{t}) = \frac{1}{q}\left(-\frac{1}{\log\rho}+\sum_{m=0}^{\infty}\frac{q\rho^m}{(1-q\rho^m)^2}+\frac{(\rho/q)\rho^m}{(1-(\rho/q)\rho^m)^2}\right).$$

Here $q=z\bar{t}$, $\rho=r^2$, $0<\rho<|q|<1$.

In the above formula we note that the expression in parentheses is real for q on the real axis and for q on the distinguished line $|q|=r$, since then $\rho/q=\bar{q}$. Furthermore, it is a continuous function of q, positive for positive q, and for $\rho<e^{-4}$ negative in a neighborhood of $q=-1$. To see the latter, denote the expression in parentheses by $\phi(q)$. We have

$$\phi(-1) = \frac{-1}{\log\rho}-\sum_{m=0}^{\infty}\frac{\rho^m}{(1+\rho^m)^2}-\sum_{m=0}^{\infty}\frac{\rho^{m+1}}{(1+\rho^{m+1})^2}\leqq-\frac{1}{\log\rho}-\frac{1}{4}.$$

The right-hand side is negative for $-\log\rho>4$, i.e., for $\rho<e^{-4}$. We conclude that $\phi(q)$ must vanish in some interior point of the q-annulus.

However, in case of simply-connected domains in C^n the question of validity of the Lu Qi-Keng conjecture still seems to be open.

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