THE TOWER THEOREM FOR FINITE GROUPS

EUGENE SCHENKMAN[1]

This note aims to give a simple proof of Wielandt's theorem that the tower of automorphisms of a finite group with center 1 ends after a finite number of steps (cf. [2] and [4, p. 245]).

Our notation is as follows: $B \lhd G$, $A \lhd \lhd G$, denote respectively that B is invariant and A subvariant in G; $\mathfrak{C}_B(A)$ and $\mathfrak{N}_B(A)$ denote respectively the centralizer and normalizer of A in B; $\mathfrak{A}(A)$ is the automorphism group of A, $|A|$ the cardinality of A, and A^ω is the smallest normal subgroup of A with A/A^ω nilpotent. All groups are finite.

Our proof of the tower theorem is based on the following facts.

(1) If $\mathfrak{C}_A(A)=1$ then $C_{\mathfrak{A}(A)}(A)=1$.

(2) If $A=A_0 \lhd A_1 \lhd \cdots \lhd A_r=G$ and if for $i=0, 1, \cdots, r-1$, $\mathfrak{C}_{A_{i+1}}(A_i)=1$, then $\mathfrak{C}_G(A)=1$ (cf [2, p. 244]).

(3) If $A \lhd \lhd G$ and if $\mathfrak{C}_G(A)=1$, then $\mathfrak{C}_G(A^\omega) \leqq A^\omega$ (cf. [1]).

(4) If $A \lhd \lhd G$ then $A^\omega B=BA^\omega$ for any $B \lhd \lhd G$ (cf [3]).

As a corollary of (4) we have the following

(4*) If $H \lhd G$ and $A \lhd \lhd G$, then $HA^\omega B=BHA^\omega$.

We also will want the following readily proven fact.

(5) If $A \lhd \lhd AB$ and if $B/B\cap A$ is simple then $A \lhd AB$.

With these results we can now prove the following result and then the tower theorem will be a direct consequence.

THEOREM. *Let $A \lhd \lhd G$ and suppose that $\mathfrak{C}_G(A)=1$, then $|G|$ is bounded in terms of $|A^\omega|$.*

PROOF. Let $\mathcal{S}_1$ denote the set of simple subvariant subgroups B_1 of G and for $i=1, 2, \cdots$, let $\mathcal{S}_{i+1}$ denote the set of subinvariant subgroups B_{i+1} of G such that B_{i+1} contains as a normal subgroup a B_i of $\mathcal{S}_i$ with B_{i+1}/B_i simple. For each i, all the subgroups B_i of $\mathcal{S}_i$ generate a normal subgroup H_i of G and we let K_i denote H_iA^ω (with K_0 denoting A^ω and H_0 denoting 1). Since $\mathfrak{C}_G(A)=1$, $A^\omega \neq 1$ and hence $H_0 \neq K_0$. Let n be minimal so that $H_n=K_n$ (n is at most the length of a composition series of A^ω). Then for $i=0, \cdots, n-1$, $K_i \lhd K_iB_{i+1}$ for any $B_{i+1} \in \mathcal{S}_{i+1}$ by (5). Hence B_{i+1} and consequently $H_{i+1} \leqq \mathfrak{N}_G(K_i)$. Then in view of (3) and the fact that $K_i=H_iA^\omega$, $|K_i| \leqq |A^\omega| \cdot |\mathfrak{A}(K_{i-1})|$ for $i=1, 2, \cdots$ and hence in particular $|K_n|$ is bounded in terms of $|A^\omega|$. But $K_n=H_n$ is normal in G and since

Received by the editors December 12, 1968.

[1] The author is indebted to the National Science Foundation for support.

$C_G(K_n) \leqq A^{\omega}$, it follows that $|G| \leqq |A^{\omega}| \cdot |\mathfrak{A}(K_n)|$. This proves the theorem.

It follows from the theorem with (1) and (2) that the tower of automorphisms of a finite group with center 1 is finite.

References

1. Eugene Schenkman, *On the tower theorem for finite groups*, Pacific J. Math. **5** (1955), 995–998.

2. H. Wielandt, *Eine Verallgemeinerung der invarianten Untergruppen*, Math. Z. **45** (1939), 209–244.

3. ———, *Vertauschbare nachinvariante Untergruppen*, Abh. Math. Sem. Univ. Hamberg, **21** (1957), 55–62.

4. Hans Zassenhaus, *Theory of groups*, 2nd ed., Chelsea, New York, 1958.

Purdue University