D-SEMIGROUPS

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If \( G \) is a topological group, \( G_0 \) will denote the identity component.

**Definition 1.** Let \( \mathcal{C} \) denote the full subcategory of the category of locally compact abelian groups whose objects \( G \) have the property that \( G/G_0 \) is a union of compact groups.

In [3] K. H. Hofmann described those locally compact semigroups which contain a proper dense maximal subgroup whose complement is compact and thus a group. Here we describe those locally compact semigroups which contain a dense subgroup \( G \in \mathcal{C} \) whose complement is a group.

If \( S \) is a topological semigroup, \( E(S) \) will denote the set of idempotents of \( S \) and \( A^* \) will denote the closure of \( A \) in \( S \) where \( A \subseteq S \). We use \( R^n \) to denote the real \( n \)-dimensional vector group.

**Definition 2.** A topological semigroup will be called a \( D \)-semigroup if \( S \) satisfies the following hypotheses:

(i) \( S \) is a locally compact Hausdorff semigroup.
(ii) \( E(S) \) contains at least two elements 1 and \( e \).
(iii) \( H(1)^* = S \).
(iv) \( eH(1) \) is a topological group.

**Definition 3.** A Hausdorff semigroup \( S \) will be said to be \( H \)-closed if \( S \) contained in a Hausdorff semigroup \( T \) as a subsemigroup implies \( S \) is a closed subspace of \( T \).

The following theorem is a result of K. H. Hofmann [3] along with the observation that a \( D \)-semigroup with \( H(e) \) compact is \( H \)-closed [7].

**Theorem 1.** Let \( S \) be a \( D \)-semigroup with \( H(e) \) compact and \( H(e) \cap H(1) \) a \( D \)-semigroup.

Then

(i) \( S = H(e) \cup H(1) \).
(ii) **Structure of** \( H(1) \). There is a maximal normal compact subgroup \( C \) and a subgroup \( M \), which is either a one-parameter group isomorphic to \( R \) or an infinite cyclic group, and \( H(1) = MC \), \( M \cap C = \{1\} \).
(iii) **The closure** \( M^* \) of \( M \) is a disjoint union of \( M \) and an abelian compact subgroup \( A \) of \( H(e) \) in which \( Me \) is dense.

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1 The author wishes to thank the referee for pointing out that our original assumption that \( G/G_0 \) is compact could be weakened to \( G/G_0 \) is the union of compact groups. His remarks also contributed to shorter and more elegant arguments.
For $G \in \mathcal{C}$, $\hat{G}$ will denote the group of characters of $G$. For a morphism $t: G \rightarrow H$ in $\mathcal{C}$, $t: \hat{H} \rightarrow \hat{G}$ will be the morphism defined by $t(\alpha) = \alpha i$.

**Lemma 1.** For locally compact abelian groups the following statements are equivalent:

(i) $G \in \mathcal{C}$.

(ii) $G \cong \mathbb{R}^n \times K$ where $K$ is a locally compact abelian group which is a union of compact subgroups.

(iii) $(\hat{G})_0 \cong \mathbb{R}^n$.

**Proof.** One has the general structure theorem [1, p. 389].

(I) A locally compact abelian group $G$ is topologically isomorphic with $\mathbb{R}^n \times K$, where $K$ is a locally compact abelian group containing a compact open subgroup. If $G$ is topologically isomorphic with $\mathbb{R}^n \times K_1$ and $K_1$ contains a compact open subgroup, then $m = n$.

(i)$\implies$(ii) If $G \cong \mathbb{R}^n \times K$ where $K$ is a locally compact abelian group containing a compact open subgroup, then $G/Go \cong K/K_0$ and $K_0$ is compact by (I). Thus $G/Go$ is a union of compact groups if and only if $K/K_0$ is a union of compact groups if and only if $K$ is a union of compact groups.

(ii)$\implies$(iii) For an abelian locally compact group $K$, $K$ is a union of compact groups if and only if it is a totally disconnected [1, 383]. Thus $(\hat{G})_0 \cong \mathbb{R}^n$ if and only if $G \cong \mathbb{R}^n \times K$ where $K$ is a union of compact subgroups.

**Lemma 2.** In the category of locally compact abelian groups $\mathcal{C}$ is closed under epics.

**Proof.** Let $i: H \rightarrow G$ be an epic in the category of locally compact abelian groups with $H \in \mathcal{C}$. Then $t: \hat{H} \rightarrow \hat{G}$ is a monic, thus $t$ is injective. Also, $t[(\hat{G})_0] \subseteq (\hat{H})_0 \cong \mathbb{R}^n$. Therefore $(\hat{G})_0 \cong \mathbb{R}^n$ and $G \in \mathcal{C}$ by Lemma 1.

**Lemma 3.** Let $i: G \rightarrow \mathbb{R}^n$ be an epic in $\mathcal{C}$. Then $i$ has a right inverse.

**Proof.** The morphism $t: \hat{\mathbb{R}^n} \rightarrow \hat{G}$ is monic, thus injective. Also, $\hat{\mathbb{R}^n} \cong \mathbb{R}^n$ and $i(\hat{\mathbb{R}^n}) \subseteq (\hat{G})_0 \cong \mathbb{R}^n$. Thus $i(\hat{\mathbb{R}^n})$ splits in $(\hat{G})_0$, and $(\hat{G})_0$ splits in $\hat{G}$ by (I). Thus $t$ has a left inverse, and $i$ has a right inverse.

**Lemma 4.** If $i: G \rightarrow H$ is an epic in $\mathcal{C}$ and $G/Go$ is compact, then $H/Go$ is compact.

**Proof.** By (I) $G \cong \mathbb{R}^n \times K$ with a $K$ a compact group. Hence, $\hat{G} \cong \mathbb{R}^n \times \hat{K}$ with $\hat{K}$ discrete [1, 362]. Since $t$ is monic, $i$ is injective. By (I) and since $t^{-1}[(\hat{G})_0]$ is a closed subgroup of $\hat{H}$, $t^{-1}[(\hat{G})_0] \cong \mathbb{R}^n$
$\times K_1$ with $K_1$ totally disconnected. Since $t$ is injective, $K_1$ is discrete. Since $(\hat{G})_0$ is open in $\hat{G}$ it follows that $(\hat{H})_0$ is open in $\hat{H}$. Thus $H \cong R^p \times C$ with $C$ a compact group.

If $S$ is a $D$-semigroup, $\phi$ will denote the map from $S$ to $eS$ defined by $\phi(s) = es$, $\Psi$ will denote the map from $H(1)$ to $H(e)$ defined by $\Psi(g) = eg$.

**Theorem 2.** Let $S$ be a $D$-semigroup with $H(1) \subseteq C$. Then

(i) $H(e) \subseteq C$.

(ii) $H(1)$ contains a locally compact subgroup $N$ and a vector subgroup $V$ such that the closure $(eN)^\ast_{H(1)}$ of $eN$ in $H(e)$ is a union of compact groups, $eV \cong V$, $V \cap N = \{1\}$, $H(1) = VN$, $eV \cap (eN)^\ast_{H(1)} = \{e\}$, $H(e) = V(eN)^\ast_{H(1)}$, and $N^\ast \cap H(e) = (eN)^\ast_{H(1)}$.

(iii) $H(1)$ contains a one-parameter group $P$ which is topologically isomorphic to $R$ and $(eP)^\ast$ is compact.

**Proof.** (i) Since $eH(1) \subseteq H(e) \subseteq (eH(1))^\ast$, $H(e)$ is a topological group [2]; thus $H(e)$ is locally compact [8]. Since $H(1)^\ast = S$, the morphism $\Psi : H(1) \rightarrow H(e)$ is an epic; thus $H(e) \subseteq C$ by Lemma 2.

(ii) By (I) and (i) there is a splitting surjection $\pi : H(e) \rightarrow W$ onto a vector group such that $\ker \pi$ is a union of compact subgroups. The morphism $\pi \Psi : H(1) \rightarrow W$ is epic in $C$ and thus has a right inverse $i : W \rightarrow H(1)$ by Lemma 3. Let $V = i(W)$ and $N = \Psi^{-1}(\ker \pi)$. Let $r_1 : H(1) \rightarrow V$ be the corestriction of $i \Psi$ to its image; thus $r_1|_V = 1_V$.

Since $s = r_1(s)|_{r_1(s)^{-1}s}]$, $H(1) = VN$. Also, it follows that $eV \cong V$, $V \cap N = \{1\}$, $(eV) \cap (eN)^\ast_{H(1)} = \{e\}$, $H(e) = (eV)(eN)^\ast_{H(1)}$, and $(eN)^\ast \cap H(e) = \ker \pi$.

(iii) By (I) $\ker \pi$ contains a compact open (in $\ker \pi$) subgroup $C$. Then $\phi^{-1}(C)$ is a locally compact abelian semigroup which contains a compact kernel $C$. Thus there is an open subsemigroup $T$ of $\phi^{-1}(C)$ such that $C \subseteq T$ and $1 \in T$ [5, 115]. Thus if $g \in T \cap H(1)$, then $1 \in K(g) = \bigcap_{n=1}^\ast \{g^i \mid i \geq n\}^\ast$. If $\Psi^{-1}(C)$ is a union of compact groups, then for all $g \in \Psi^{-1}(C) 1 \in K(g)$. Thus either $T \cap H(1) = \emptyset$ or $\Psi^{-1}(C)$ contains a one-parameter group isomorphic to $R$. Since $e \in (\Psi^{-1}(C))^\ast$, $T \cap H(1) \neq \emptyset$; thus $\Psi^{-1}(C)$ contains a one-parameter group $P$ isomorphic to $R$. By Weil's lemma [6, 102], $(eP)^\ast$ is compact.

**Corollary.** Let $S$ be a locally compact Hausdorff semigroup which contains a dense group $H(1) \subseteq C$. If $H(1)$ is a union of compact subgroups, then either $E(S) = \{1\}$ or for all $e \in E(S) \setminus \{1\}$, $H(e)$ is not a topological group.

**Theorem 3.** Let $S$ be a $D$-semigroup with $H(1) \subseteq C$ and $S \setminus H(1) = H(e)$. Then there is a vector subgroup $V \subseteq H(1)$ and a subsemigroup $T$ of $S$
such that:

(i) The function \( m: V \times T \rightarrow S \) defined by \( m(v, t) = vt \) is an isomorphism of topological semigroups.

(ii) \( T \) is a D-semigroup with \( H_T(e) = T \setminus H_T(1) \).

(iii) \( H_T(1) = H_S(1) \cap T \cong R \times K \) where \( K \) is a union of compact open subgroups of \( K \), and \( H_T(e) = H_S(e) \cap T \) is a union of compact open subgroups of \( H_T(e) \).

If in addition, \( H_S(1)/H_S(1)_e \) is compact, then \( T \) is one of the semigroups described by Hofmann in [3].

Proof. Let \( T = \phi^{-1}(\ker \pi) \). Then \( T \cap H(e) = H_T(e) = \ker \pi \) is a union of compact open subgroups of \( H_T(e) \). Let \( r \) be the corestriction of \( \iota \phi \) to its image. Then \( r \mid V = 1 \). The morphism \( s \rightarrow (r(s), r(s)^{-1}) \): \( S \rightarrow V \times T \) is the inverse of \( m \) in (i). Part (ii) follows from Theorem 2. Finally, let \( C \) be a compact open subgroup of \( T \cap H(e) \). Then \( \phi^{-1}(C) \) is open in \( T \) and is a D-semigroup, but in addition \( \phi^{-1}(C) \cap H(e) \) is compact. Thus by Hofmann's results [3], \( \phi^{-1}(C) \cong R \times K \) with \( K \) a compact group. Since \( \phi^{-1}(C) \cap H(1) \) is open in \( T \cap H(1) \), from (I) it follows that \( T \cap H(1) \cong R \times K_1 \) where \( K_1 \) is a union of compact subgroups. Since \( H_T(e) = \ker \pi \), we have (iii).

Now assume \( H(1)/H(1)_e \) is compact. Then \( H_T(1)/H_T(1)_e \) is compact, hence, by Lemma 4 and (iii), \( H(e) \cap T = H_T(e) \) is compact.

Theorem 4. Let \( S \) be a D-semigroup with property that \( H(1) \) is a topological group which contains a compact normal subgroup \( C \) and \( H(1)/C \subset C \). Then there is a D-semigroup \( S/C \) and an open homomorphism \( h \) from \( S \) onto \( S/C \) such that:

(i) \( h(H(1)) = H(h(1)) \cong H(1)/C \).

(ii) \( S/C = [H(h(1))]^* \).

(iii) \( h[H(e)] = H(h(e)), h^{-1}[h(H(e))] = H(e), \) and \( H(h(e)) \cong H(e)/eC \).

The proof of this theorem presents no difficulties and is left to the reader.

Corollary 1. Let \( S \) be a locally compact Hausdorff semigroup satisfying the following hypotheses:

(a) \( E(S) \) contains at least two elements \( 1 \) and \( e \).

(b) \( H(1) \) is a topological group such that \( H(1)^* = S \).

(c) \( H(1) \) contains a compact normal subgroup \( C \) such that \( H(1)/C \cong R \).

Then the boundary of \( H(1) \) is a compact group if and only if \( eH(1) \) is a topological group.

Proof. If the boundary of \( H(1) \) is compact, then by the results of Hofmann [3] \( S \setminus H(1) \) is a compact group.

If \( eH(1) \) is a topological group, then \( H(e) \) is a topological group.
By Theorem 4 \( S/C \) is a \( D \)-semigroup with \( h(H(1)) \cong R \). By Theorem 2 \( H(h(e)) \) is compact; thus by a result of J. G. Horne [6], \( S/C = H(h(1)) \cup H(h(e)) \). Thus \( S = H(1) \cup H(e) \) by Theorem 3. Since \( H(e)/eC \) is compact and \( C \) is compact, \( H(e) \) is compact.

**Corollary 2.** Let \( S \) be as in Theorem 4 with the additional property that \( H(h(1))/H(h(1)) \) is compact. Then \( H(e) \cup H(1) \) is a \( D \)-semigroup if and only if \( h(h(e)) \cong \mathbb{R}^{n-1} \times C \) where \( C \) is compact and \( H(h(1)) \cong \mathbb{R}^n \times K \) with \( K \) compact.

**Proof.** Let \( S/C \) be the semigroup described in Theorem 3. Then \( H(e) \cup H(1) \) is a \( D \)-semigroup if and only if \( H(h(e)) \cup H(h(1)) \) is a \( D \)-semigroup. If \( H(h(e)) \cup H(h(1)) \) is a \( D \)-semigroup, then \( h(H(e) \cup H(1)) \) satisfies the conditions of Theorem 2. The only if part now follows from Theorem 2.

Assume \( H(e) \cong \mathbb{R}^{n-1} \times C \). By (1) there is a splitting surjection \( \pi \) from \( H(h(e)) \) onto a vector group such that \( \ker \pi \) is compact. Then \( \phi^{-1}(\ker \pi) \) is a \( D \)-semigroup with a compact ideal \( \ker \pi \) and \( \phi^{-1}(\ker \pi) \cap H(h(1)) \cong \mathbb{R} \times K \) where \( K \) is compact. Thus by Corollary 1 \( \phi^{-1}(\ker \pi) = \ker \pi \cup (\phi^{-1}(\ker \pi) \cap H(h(1))) \), and \( \phi^{-1}(\ker \pi) \) is a closed subspace of \( S/C \). By Lemma 3 the morphism \( \pi \Psi: H(h(1)) \to W \) has a right inverse \( i: W \to H(h(1)) \). Let \( V = i(W) \) and \( r: H(e) \cup H(1) \to V \) be the corestriction of \( i \pi \phi \) to its image. Let \( m: V \times \phi^{-1}(\ker \pi) \to h(H(1) \cup H(e)) \) be the map defined by \( m(v, t) = vt \). Then

\[
s \mapsto (r(s), r(s)s^{-1}): h(H(1) \cup H(e)) \to V \times \phi^{-1}(\ker \pi)
\]

is the inverse of \( m \). Thus \( h(H(1) \cup H(e)) \) is a \( D \)-semigroup.

**References**


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