ABOUT THE UNIVERSAL COVERING OF THE COMPLEMENT OF A COMPLETE QUADRILATERAL[1]

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In 1960, Professor Chern asked me if the universal covering of the complement of a complete quadrilateral Q in the two-dimensional complex projective space $\boldsymbol{P}^2$ were biholomorphic equivalent to a bounded domain in $\boldsymbol{C}^2$. The answer is negative. A recent discussion in Berkeley disclosed that this fact still seems to be unknown. Therefore, the simple proof shall be finally published. At first, a more general theorem shall be proved:

THEOREM 1. *Let M be a complex space. Let $\pi: \hat{M} \to M$ be an unrestricted covering space of M. Let $g: \hat{M} \to \boldsymbol{C}^n$ be a holomorphic map of fiber dimension zero. Suppose that a nonconstant holomorphic map $f: \boldsymbol{C} - \{0\} \to M$ exists. Then, the image $g(\hat{M})$ of g is unbounded.*

REMARKS TO THE TERMINOLOGY. $\pi: \hat{M} \to M$ is called a covering if π is surjective and locally biholomorphic. The covering is called unrestricted (or regular by Ahlfors and Sario) if, for every $a \in M$, every $b \in \pi^{-1}(a)$ and every continuous map $\gamma: I \to M$ of the unit interval I into M with $\gamma(0) = a$, a continuous map $\tilde{\gamma}: I \to \hat{M}$ with $\tilde{\gamma}(0) = b$ and $\pi \circ \tilde{\gamma} = \gamma$ exists, i.e. if the curves of M lift into $\hat{M}$. The holomorphic map g is said to be of fiber dimension zero if the inverse images $g^{-1}(y)$ consist of isolated points only, or are empty.

PROOF OF THEOREM 1. Let $\epsilon: \boldsymbol{C} \to \boldsymbol{C} - \{0\}$ be the exponential map: $\epsilon(z) = e^z$. Because $\boldsymbol{C}$ is simply connected, a holomorphic map $h: \boldsymbol{C} \to \hat{M}$ exists such that $\pi \circ h = f \circ \epsilon$. Suppose that $g(\hat{M})$ is bounded. Then $g \circ h$ is a bounded holomorphic vector function on $\boldsymbol{C}$. By Liouville's theorem, $g \circ h$ is constant. Hence, a point $c \in \boldsymbol{C}^m$ exists such that the connected set $h(\boldsymbol{C})$ is contained in $g^{-1}(c)$, whose components are points. Therefore, h is constant. Hence $f \circ \epsilon = \pi \circ h$ is also constant. Because ϵ is surjective, f is constant. Contradiction: Q.E.D.

THEOREM 2.[2] *Let Q be the union of 4 projective lines in general position in the two-dimensional complex projective space $\boldsymbol{P}^2$. Then the universal*

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[2] ADDED IN PROOF. In the meanwhile, I was informed that Peter Kiernan, Berkeley, has proved the same theorem, which will be published in these Proceedings.

covering $\pi: \hat{M} \to M$ *of* $M = \boldsymbol{P}^2 - Q$ *is not biholomorphically equivalent to an open bounded subset of* $\boldsymbol{C}^2$.

PROOF. Suppose a biholomorphic map $g: \hat{M} \to H$ onto an open bounded subset H of $\boldsymbol{C}^2$ exists. Let L be a diagonal of Q. In an appropriate coordinate system, M and $M \cap L$ are given by

$$M = \{(z, w) \in \boldsymbol{C}^2 \mid zw(z + w - 1) \neq 0\},$$

$$f: \boldsymbol{C} - \{0\} \to L \cap M \text{ biholomorphically by } f(z) = (z, -z).$$

Hence, $f: \boldsymbol{C} - \{0\} \to M$ is not constant. Theorem 1 implies that $g(\hat{M}) = H$ is not bounded. Contradiction: Q.E.D.

Of course, the proof shows that any unrestricted covering of $\boldsymbol{P}^2 - Q$ cannot be spread as a ramified covering space over a bounded subset of any $\boldsymbol{C}^m$. However, Chern's original conjecture becomes true if a diagonal is removed:

PROPOSITION 3. *The dicylinder is the universal covering of* $A = \boldsymbol{P}^2 - (Q \cup L)$, *where* L *is a diagonal of the quadrilateral* Q.

PROOF. Observe that $E = \{z \in \boldsymbol{C} \mid z \neq 0, 1\}$ has the unit disk $D = \{z \mid |z| < 1\}$ as universal covering. For an appropriate coordinate system is

$$A = \{(z, w) \in \boldsymbol{C}^2 \mid zw(w + z - 1)(w + z) \neq 0\}.$$

A biholomorphic map $f: A \to E \times E$ is defined by

$$f(z, w) = (1/(z + w), z/(z + w)).$$

Therefore, $D \times D$ is the universal covering of A. Q.E.D.

The question of whether an open subset of $\boldsymbol{C}^2$ is a universal covering of $\boldsymbol{P}^2 - Q$ remains unsettled. (See Problem 25, p. 308, *Proceedings of the conference on complex analysis* (Minneapolis 1964), Springer-Verlag, Berlin, 1965.)

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