ABOUT THE UNIVERSAL COVERING OF THE COMPLEMENT OF A COMPLETE QUADRILATERAL

WILHELM STOLL

In 1960, Professor Chern asked me if the universal covering of the complement of a complete quadrilateral \( Q \) in the two-dimensional complex projective space \( \mathbb{P}^2 \) were biholomorphic equivalent to a bounded domain in \( \mathbb{C}^2 \). The answer is negative. A recent discussion in Berkeley disclosed that this fact still seems to be unknown. Therefore, the simple proof shall be finally published. At first, a more general theorem shall be proved:

**Theorem 1.** Let \( M \) be a complex space. Let \( \pi: \tilde{M} \to M \) be an unrestricted covering space of \( M \). Let \( g: \tilde{M} \to \mathbb{C}^n \) be a holomorphic map of fiber dimension zero. Suppose that a nonconstant holomorphic map \( f: \mathbb{C} \to \{0\} \to M \) exists. Then, the image \( g(M) \) of \( g \) is unbounded.

**Remarks to the terminology.** \( \pi: \tilde{M} \to M \) is called a covering if \( \pi \) is surjective and locally biholomorphic. The covering is called unrestricted (or regular by Ahlfors and Sario) if, for every \( a \in M \), every \( b \in \pi^{-1}(a) \) and every continuous map \( \gamma: I \to M \) of the unit interval \( I \) into \( M \) with \( \gamma(0) = a \), a continuous map \( \tilde{\gamma}: I \to \tilde{M} \) with \( \tilde{\gamma}(0) = b \) and \( \pi \circ \tilde{\gamma} = \gamma \) exists, i.e. if the curves of \( M \) lift into \( \tilde{M} \). The holomorphic map \( g \) is said to be of fiber dimension zero if the inverse images \( g^{-1}(y) \) consist of isolated points only, or are empty.

**Proof of Theorem 1.** Let \( \epsilon: \mathbb{C} \to \mathbb{C} \to \{0\} \) be the exponential map \( \epsilon(z) = e^z \). Because \( \mathbb{C} \) is simply connected, a holomorphic map \( h: \mathbb{C} \to \tilde{M} \) exists such that \( \pi \circ h = f \circ \epsilon \). Suppose that \( g(\tilde{M}) \) is bounded. Then \( g \circ h \) is a bounded holomorphic vector function on \( \mathbb{C} \). By Liouville's theorem, \( g \circ h \) is constant. Hence, a point \( c \in \mathbb{C}^n \) exists such that the connected set \( h(\mathbb{C}) \) is contained in \( g^{-1}(c) \), whose components are points. Therefore, \( h \) is constant. Hence \( f \circ \epsilon = \pi \circ h \) is also constant. Because \( \epsilon \) is surjective, \( f \) is constant. Contradiction: Q.E.D.

**Theorem 2.** Let \( Q \) be the union of 4 projective lines in general position in the two-dimensional complex projective space \( \mathbb{P}^2 \). Then the universal
covering \( \pi: \hat{M} \to M \) of \( M = \mathbb{P}^2 - Q \) is not biholomorphically equivalent to an open bounded subset of \( \mathbb{C}^2 \).

**Proof.** Suppose a biholomorphic map \( g: \hat{M} \to H \) onto an open bounded subset \( H \) of \( \mathbb{C}^2 \) exists. Let \( L \) be a diagonal of \( Q \). In an appropriate coordinate system, \( M \) and \( M \cap L \) are given by

\[
M = \{(z, w) \in \mathbb{C}^2 \mid zw(z + w - 1) \neq 0\},
\]

\[
f: C - \{0\} \to L \cap M \text{ biholomorphically by } f(z) = (z, -z).
\]

Hence, \( f: C - \{0\} \to M \) is not constant. Theorem 1 implies that \( g(\hat{M}) = H \) is not bounded. Contradiction: Q.E.D.

Of course, the proof shows that any unrestricted covering of \( \mathbb{P}^2 - Q \) cannot be spread as a ramified covering space over a bounded subset of any \( \mathbb{C}^n \). However, Chern's original conjecture becomes true if a diagonal is removed:

**Proposition 3.** The dicylinder is the universal covering of \( A = \mathbb{P}^2 - (Q \cup L) \), where \( L \) is a diagonal of the quadrilateral \( Q \).

**Proof.** Observe that \( E = \{z \in \mathbb{C} \mid z \neq 0, 1\} \) has the unit disk \( D = \{z \mid |z| < 1\} \) as universal covering. For an appropriate coordinate system is

\[
A = \{(z, w) \in \mathbb{C}^2 \mid zw(w + z - 1)(w + z) \neq 0\}.
\]

A biholomorphic map \( f: A \to E \times E \) is defined by

\[
f(z, w) = (1/(z + w), z/(z + w)).
\]

Therefore, \( D \times D \) is the universal covering of \( A \). Q.E.D.

The question of whether an open subset of \( \mathbb{C}^2 \) is a universal covering of \( \mathbb{P}^2 - Q \) remains unsettled. (See Problem 25, p. 308, *Proceedings of the conference on complex analysis* (Minneapolis 1964), Springer-Verlag, Berlin, 1965.)

**University of Notre Dame**