CENTRAL IDEMPOTENTS IN GROUP RINGS[1]

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Let $K[G]$ denote the group ring of a finite group G over a field K of characteristic $p>0$. If $\alpha=\sum_{x\in G} a_x x\in K[G]$ we let the support of α be Supp $\alpha=\{x\in G \mid a_x\neq 0\}$. A well-known result of Osima [**2**, p. 178] gives the explicit form for the central idempotents in $K[G]$ and in particular shows that their support consists of p'-elements of G. For most applications only the latter fact is needed. The proof of this result is character theoretic in nature and essentially requires lifting $K[G]$ to a group ring over some p-adic field. In this paper we give an elementary character-free proof of

THEOREM. *Let e be a central idempotent in $K[G]$. Then* Supp e *consists of p'-elements.*

We require the following few facts:

(1) Let P be a p-subgroup of G and let s denote the natural projection $s:K[G]\to K[\mathbf{C}(P)]$. Then s induces a ring homomorphism, the Brauer homomorphism, from $\mathbf{Z}(K[G])$ into $\mathbf{Z}(K[\mathbf{C}(P)])$ [**1**, Satz 7A].

(2) Let S denote the subspace of $K[G]$ spanned by all elements of the form $\alpha\beta-\beta\alpha$ with $\alpha,\beta\in K[G]$. Then for $\alpha_1,\alpha_2,\cdots,\alpha_m\in K[G]$ we have

$$(\alpha_1+\alpha_2+\cdots+\alpha_m)^{p^n}\equiv \alpha_1^{p^n}+\alpha_2^{p^n}+\cdots+\alpha_m^{p^n} \pmod S$$

(see [**1**, Satz 3A]).

(3) Let S be as above and let x be a central element of G of order a power of p. If $\alpha\in S$ then $x\notin$ Supp α (see [**1**, Satz 3B]).

Note that (3) above is merely the simple observation that if $x, y, z\in G$ and if x is central in $K[G]$ then $x\notin$ Supp $(yz-zy)$.

We now proceed to prove the theorem. Suppose z is an element of Supp e which is not a p'-element and write $z=xy=yx$ where $x\neq 1$ has order a power of p and where q, the order of y, is prime to p. Let $P=\langle x\rangle$. Then by (1), $s(e)$ is a central idempotent in $K[\mathbf{C}(\mathrm{P})]$ and $z\in$ Supp $s(e)$. Thus it clearly suffices to assume that x is central in G.

Choose integer n with $p^n\geq |G|$ and with $p^n\equiv 1 \pmod q$ and set $\alpha=y^{-1}e$. If $\alpha=\sum_{g\in G} a_g g$ then by (2) $\alpha^{p^n}\equiv\sum(a_g)^{p^n}g^{p^n} \pmod S$.

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Now $p^n \geqq |G|$ so g^{p^n} is a p'-element and hence by (3), $x \notin \text{Supp } \alpha^{p^n}$. On the other hand since e is a central idempotent and since $p^n \equiv 1 \pmod q$ we have $\alpha^{p^n} = (y^{-1})^{p^n} e^{p^n} = y^{-1}e = \alpha$. Since, by definition of α, $x \in \text{Supp } \alpha$, this is a contradiction and the result follows.

We remark that this proof holds for group rings $R[G]$ where R is any commutative ring with 1 satisfying $pR = 0$ and it yields the same result. In fact R need not even be commutative since $1 \in R$ implies immediately $\mathbf{Z}(R[G]) \subseteq \mathbf{Z}(R)[G]$. In addition this proof will also handle the twisted group rings $K^t[G]$ once the following simple observation is made.

(4) Let Z be a central p-subgroup of G. Then $K^t[Z]$ is central in $K^t[G]$.

With this fact, (1) and (3) carry over easily to the twisted case.

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References

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